ON THE COMBINATORICS OF TABLEAUX — A SURVEY

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ABSTRACT. We survey RSK-type tableau insertion algorithms and the adjacent parts of discrete mathematics. ***

CONTENTS

1. INTRODUCTION

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2. FOMIN LATTICES

Need to list all the Fomin lattices per se and all the techniquest of constructing fomin lattices. Don't forget lattice-dual, which negates the differential degree.

See [Wor2026b, sec. 2 and 3] for a description of Fomin lattices. Sec. 2.3 contains a list of all known Fomin lattices. Th. 7.1 contains a classifiction of all Fomin lattices under certain restritions. These restrictions are satisfied by all known positive distributive Fomin lattices.

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Date: May 1, 2026.

3. Growth diagrams

See [Wor2023a, sec. 2] for a description of how Fomin lattices are used to generate growth diagrams, which implement tableau insertion algorithms.

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4. Insertion algorithms

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The currently known types of tableaux are:

(1) Young tableaux, which represent saturated chains in the lattice of partitions of integers.
(2) Shifted Young tableaux, which represent saturated chains in the lattice of strict partitions of integers. These allow two colors on the off-diagonal elements of the diagram and one color on the diagonal elements.
(3) k-row Young tableaux, which represent saturated chains in the lattice of partitions of integers with $\leq k$ parts. These allow variable numbers of colors on elments of the diagram.
(4) Young–Fibonacci tableaux, which are saturated chains in the Young–Fibonacci lattice.

Only the first two types of tableaux have been given much attention.

(add symplectic tableaux)

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5. Jeu de taquin

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6. Greene's invariant

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7. Knuth relations

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8. Serialization

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See that reference that serializes growht diagrams. Dob2926a

-rw-r–r–. 1 worley worley 181680 Apr 24 16:23 Gess2026a.pdf -rw-r–r–. 1 worley worley 287420 Apr 24 16:19 Gess1989a.pdf -rw-r–r–. 1 worley worley 201187 Apr 24 15:35 Cel2023b.pdf -rw-r–r–. 1 worley worley 1137181 Apr 24 15:34 Cel2023a.pdf

From: "Dale R. Worley" ¡Dale.Worley@comcast.net¿ To: gessel@brandeis.edu Subject: Re: Generalized RSK In-Reply-To: ¡87o6j8hyef.fsf@hobgoblin.ariadne.com¿ (Dale.Worley@comcast.net) Fcc: /Tableaux/TABLEAUX Fcc: /temp/sent-mail Date: Sun, 26 Apr 2026 09:58:47 -0400 Message-ID: ¡87se8hlt60.fsf@hobgoblin.ariadne.com¿ –text follows this line– Ira,

I realize that my previous message was incomprehensible.

¿ I've only skimmed genschur.pdf but I think it it has some bearing on the ¿ process of "sequentializing": For ordinary RSK, an input sequence of ¿ positive integers is renumbered 1 to n by numbering the subsets equal to ¿ a single integer in major order by that value and minor order ¿ left-to-right. And for shifted RSK, there is a similar sequentializing ¿ process, but for each subset equal to an integer, first you number the ¿ circled elements right-to-left and then the uncircled elements ¿ left-to-right.

There is a construction in my thesis which is an instance of the relation-pairs R and S in "Generalized Schur functions". In order to describe how both uncircled and uncircled numbers could be put into both unshifted and shifted tableaux, I had to define an order relation "leftarrow". (sec. 2.1) Generally, numbers are ordered by magnitude and circled numbers are less than the same uncircled number. But an uncircled number is leftarrow itself, while an uncircled number is not. The elements of a row of a tableau are related by leftarrow.

The elements of a column are related (as "Generalized Schur functions" predicts) by the reverse-complement of leftarrow, which I denoted double-leftarrow.

Leftarrow is the interleaving of two total orders, the circled numbers, which are irreflexive, and the uncircled numbers, which are reflexive.

But as I was trying to describe in the previous message, this order has ties to how you "sequentialize". That concept is: Given a sequence of numbers that are input to the Knuth algorithm to create a semi-standard tableau. "Sequentializing" is to replace those numbers with 1 ... n in the correct way, input those to the Schensted algorithm to create a standard tableau, then replace 1 ... n in the standard tableau with the original elements they replaced to make a semi-standard tableau. This should produce the semi-standard tableau produced directly by the Knuth algorithm.

In the unshifted case without circles, the numbers are sequentialized by numbering the elements in increasing order, with ties resolved going from left to right. With circles, the numbers are sequentialized by numbering the elements in increasing order, with circled numbers before the corresponding uncircled numbers, but with equal circled numbers numbered right-to-left and equal uncircled numbers numbered left-to-right.

I have done no investigation, but I suspect that that difference between uncircled numbers and circled numbers is derived from the fact that under leftarrow uncircled numbers are reflexive and circled numbers are irreflexive. This probably generalizes to all the situations covered by "Generalized Schur functions".

Dale

9. Hook-length formula

*** NguyenVulWood2025a, Pak2001a, Proct1999a, companion paper to Proct199a

10. Representation of towers of algebras

*** Gaetz paper reference. Give a lattice, is the tower unique?

11. Group representations

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The towers of initial interest are: Sn (the symmetric groups) An (the alternating groups) Bn (the hyperoctahedral) groups The Schur cover of Sn.

We know that the representations of Sn are indexed by the partitions, organized as Young's lattice.

For Bn, the representations are indexed by pairs of partitions whose size sums to n. From [WikiHyp]:

"Concretely, the isomorphism classes of irreducible representations of B_n are indexed by pairs (λ, μ) of integer partitions for which the sum $|\lambda| + |\mu|$ of the sizes is equal to n. The irreducible character $\chi_{\lambda\emptyset}$ indexed by $(\lambda, \emptyset)$ is the induced representation of the symmetric group character indexed by λ from the subgroup S_n to B_n; the character $\chi_{\emptyset\mu}$ indexed by $(\emptyset, \mu)$ is the tensor product $\delta \otimes \chi_{\mu\emptyset}$ where δ the is linear character that takes the value +1 on transpositions in B_n and takes the value -1 on sign-change reflections. If λ is a partition of k and μ is a partition of $n - k$, then $\chi_{\lambda\mu}$ is the induction product of $\chi_{\lambda\emptyset} \otimes \chi_{\emptyset\mu}$ from $B_k \times B_{n-k}$ to B_n. These characterizations can be combined with the Murnaghan–Nakayama rule to give a combinatorial formula for character values. And there is a characteristic map from the character ring to a ring of symmetric functions in two sets of variables."

This is footnoted: Stembridge, John R. (1992), "The projective representations of the hyperoctahedral group", J. Algebra, 145 (2): 396–453, doi:10.1016/0021-8693(92)90110-8, hdl:2027.42/30235

These are related to the lattice $\mathbb{Y}^2$, but I haven't worked out the details. There is an interesting complication that B_1 is not the trivial group, so embedding $B_n \times B_1$ into B_{n+1} is not an ebedding of B_n into B_{n+1}.

Regarding A_n, the alternating groups, Gemini says: The representation theory of alternating groups A_n focuses on classifying the complex irreducible representations (irreps) of the group of even permutations. It is intimately linked to the representation theory of symmetric groups S_n via Clifford theory. Irreps of A_n correspond to partitions of n, where non-self-conjugate partitions remain irreducible upon restriction from S_n, and self-conjugate partitions split into two distinct irreps of the same dimension of A_n.

Regarding the projective representations of S_n Gemini says: The irreducible projective representations of S_n are classified by strict partitions of n, often labeled using shifted Young tableaux.

So the natural tower for $\mathbb{SY}$ is the Schur covers of S_n. Need to sort out the Hopf algebra of that (that is, the product formulas and by implication the splitting).

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Mention wreath products.

Catalog what is known about the repr theory of these groups, including restriction/induction. And also product, which may be more important.

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Start with Bump and Schilling re crystals.

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Davies, J. W.; Morris, A. O. (1974), "The Schur Multiplier of the Generalized Symmetric Group", J. London Math. Soc., 2, 8 (4): 615–620, doi:10.1112/jlms/s2-8.4.615 Can1996a Can, Himmet (1996), "Representations of the Generalized Symmetric Groups", Contributions to Algebra and Geometry, 37 (2): 289–307, CiteSeerX 10.1.1.11.9053 Only G(m, 1, n). Osima, M. (1954), "On the representations of the generalized symmetric group", Math. J. Okayama Univ., 4: 39–54

MorJon2003a.pdf label = MorJon2003, author = Morris, Alun Owen, author = Jones, Huw I., title = Projective Representations of Generalized Symmetric Groups, branching rules

ChatGPT query: What is known of the branching rules for the generalized symmetric groups?

5. References (classical sources) Ariki–Koike (representations of Hecke algebras of G(m,1,n)) James–Kerber (for wreath products) Geck–Pfeiffer (finite Coxeter groups and related structures) Broué–Malle–Rouquier (complex reflection groups)

11.1. Motivation. Regarding group representations, we are guided by the example of the tower of symmetric groups S_n. For every n, S_n is a subgroup of S_{n+1}, so every irrep of S_n induces a representation on S_{n+1} and has a restriction to S_{n-1}. The splitting diagram of these representations is the lattice of partitions $\mathbb{Y}$, which drives growth diagrams for the R–S algorithm. Hence we expect the representations of other towers of groups to generate the lattices which drive other instances of growth diagrams.

Less obviously, for every n and m, $S_n \times S_m$ can be embedded into S_{n+m}. Given irreps of S_n and S_m, the splitting of the induced representation of S_{n+m} gives the Littlewood–Richardson coefficients, which are the structure coefficients for multiplying the corresponding symmectric functions. So we expect any useful tower of groups to have the same property.

For any tower of groups G_n that satisfies the above product property, G_n is isomorphic to $G_n \times \{e_{G_1}\}$, which is embeddable into $G_n \times G_1$, which is embeddable into G_{n+1} by the product property. Thus the $G_\bullet$ have the chain embedding property that is the usual definition of a tower of groups. So the product property is strictly stronger than the chain embedding property.

For the example of $S_\bullet$, we can define S_0 to be the trivial group, which has one representation operating on the 0-dimensional vector space. And trivially, $S_n \times S_0$ can be embedded into S_n, so adding S_0 to the tower preserves the product property. Thus we expect any tower of groups to have a G_0 which is the trivial group.

11.2. Complexified symmetric groups. A natural place to start is the complexified symmetric groups, also called complex reflection groups.[WikiComp] We define $C(n; k, r)$ to be the group of $n \times n$ complex matrices with exactly one non-zero entry in each row and column, all non-zero entries are k-th roots of unity, and the product of all the non-zero entries is a k/p-th root of unity. Thus we require that p divides k. $C(n; k, 1)$ is the generalized symmetic group, which has no additional restriction on the product of the non-zero entries. For any k and r, we can define $C(0; k, r)$ to be the trivial group realized as the 0×0 complex matrix.

The complexified symmetric groups can be organized into towers with the chain embedding property in many ways. The usual way is to fix k and r: $G(0; k, r), G(1; k, r), G(2; k, r), \ldots$. But towers can be built by varying k: $G(n; 1, 1), G(n; 2, 1), G(n; 2^2, 1), \ldots$, and various more complex towers can be built. But if we demand the product property, the natural towers are the ones that fix k and r. Thus we write the symbol $G(n; k, r)$ with a semicolon to distinguish the parameter n (which we think of as variable) and the parameters k and r (which we think of as fixed).

[WikiComp] notes that all irreducible complex reflection groups are either $G(n; k, r)$ or one of 34 exceptional groups. Thus in regard to building towers, the $G(n; k, r)$ are the only complex reflection groups to be considered. The complex reflection groups naturally contain all of the real reflection (Coxeter) groups, so we do not need to give any special consideration to the Coxeter groups.

`https://math.stackexchange.com/questions/5132822/the-tower-of-symmetric-groups` What remains: Does Brandeberg's construction help us with other towers? Does B automagically give associativity? He's kinda vague about that. Are the maps Sn x Sm unique up to automorphisms? Reference for uniqueness of Sn+1 embedding.

Write question about SnxSm embedding.

`https://en.wikipedia.org/wiki/Automorphisms_of_the_symmetric_and_alternating_groups`

All automorphisms of S_n are conjugation, except that $\text{Aut}(S_6)/\text{Inner}(S_6) = C_2$ because there is a special automorphism of S_6. It has something to do with the symmetries of the icosahedron. See the Wikipedia article on the automorphisms of the symmetric group.

ChatGPT says that embeddings of S_n into Sn+1 are unique up to automorphisms.

ChatGPT says that embeddings of $S_n \times S_m$ into Sn+m are unique up to automorphisms. In all ordinary cases, all automorphsms are conjugations, so S_n acts on some subset of n+m and Sm acts on its complement.

If $n = m$ there is a source automorpism that swaps the two arguments, but that maps through into swapping two blocks of $n = m$ elements of $n + m$ as long as $n + m \neq 6$. But if $n + m = 6$, the natural embedding followed by an outer automorphism is likely to be messy, and swapping the two arguments may not map into a conjugation in S_6.

Consider the following structure: For every $n, m > 0$, choose an embedding f_{nm} from $S_n \times S_m$ into S_{n+m}. Given any $n, m, p > 0$, we can use the f's to construct an embedding of $S_n \times S_m \times S_p$ into S_{n+m+p} in two ways: One way is to apply f_{nm} to S_n and S_m and then apply $f_{n+m,p}$ to S_{n+m} and S_p to get a result in S_{n+m+p}. The other way is to apply f$_{mp}$ to the last two factors first and then apply $f_{n,m+p}$ to get a result in S_{n+m+p}. Call the structure, or really, the set of f's, "associative" if these two embeddings are identical for all n, m, p. My first question is whether there is a name for this sort of structure (replacing S_n with a general series of groups G_n)? Particularly, is there a term I can search for in the mathematical literature to find what is already known about this sort of structure?

What you're describing is a categorical/algebraic structure related to monoidal categories and symmetric operads.

This is strict associativity at the level of embeddings.

A graded monoid in the category of groups: Your G_n are the graded pieces. The f_{nm} are the multiplication maps. Associativity is exactly the commutativity of the diagram you drew.

Or, equivalently, it can be phrased in terms of non-symmetric operads in the category of groups: In operad language, G_n is the n-th "arity" group. Composition maps satisfy associativity.

Step 3: Names you can search for Graded monoid in groups (or "graded associative system of groups"). Non-symmetric (or plain) operad in groups. Symmetric operad if you include actions of S_n compatible with f_{nm}. Sometimes also called "species of groups with substitution" in combinatorial literature. In the literature: "Graded monoid in a category" is a very standard reference. For example, a graded monoid in Sets gives a "composition law" similar to your f_{nm}. "Operad in groups" appears in algebraic topology and higher category theory contexts (e.g., May's The Geometry of Iterated Loop Spaces).

now we are stepping into a rigid-structure question about graded associative embeddings of symmetric groups

This is exactly asking whether all associative systems of embeddings of S_n are equivalent up to automorphisms of each S_n.

11.3. Equivalences with other towers of groups. From [WikiComp]:

It uses the notation $G(m, p, n)$, where the matrices are $n \times n$, with all matrix entries being m-th roots of unity and the product of the entries being a m/p-th root of unity.

The group $G(m, 1, n)$ is the generalized symmetric group; equivalently, it is the wreath product of the symmetric group S_n by a cyclic group of order m: $G(m, 1, n) = C_m \wr_{[n]} S_n$ where S_n is acting naturally on the set $[n]$.

The group $G(m, p, n)$ is an index-p subgroup of $G(m, 1, n)$. $G(m, p, n)$ is of order $m^n n!/p$. As matrices, it may be realized as the subset in which the product of the nonzero entries is an m/p-th root of unity (rather than just an m-th root). Algebraically, $G(m, p, n)$ is a semidirect product of an abelian group of order m^n/p by the symmetric group Sym(n); the elements of the abelian group are of the form $(\theta^{a_1}, \theta^{a_2}2, \ldots, \theta^{a_n})$, where θ is a primitive mth root of unity and $\sum a_i \equiv 0 \bmod p$, and Sym(n) acts by permutations of the coordinates.

$G(1, 1, n)$ has type $A_{n-1} = [3, 3, \ldots, 3, 3]$; the symmetric group of order $n!$

$G(2, 1, n)$ has type $B_n = [3, 3, \ldots, 3, 4]$; the hyperoctahedral group of order $2^n n!$

$G(2, 2, n)$ has type $D_n = [3, 3, \ldots, 31, 1]$ with order $2^n n!/2$.

In addition, when $m = p$ and $n = 2$, the group $G(p, p, 2)$ is the dihedral group of order $2p$; as a Coxeter group, type $I_2(p) = [p]$ (and the Weyl group G_2 when $p = 6$).

Other special cases and coincidences

The only cases when two groups $G(m, p, n)$ are isomorphic as complex reflection groups are that $G(ma, pa, 1)$ is isomorphic to $G(mb, pb, 1)$ for any positive integers a, b (and both are isomorphic to the cyclic group of order m/p). However, there are other cases when two such groups are isomorphic as abstract groups.

The groups G(3, 3, 2) and G(1, 1, 3) are isomorphic to the symmetric group Sym(3). The groups G(2, 2, 3) and G(1, 1, 4) are isomorphic to the symmetric group Sym(4). Both G(2, 1, 2) and G(4, 4, 2) are isomorphic to the dihedral group of order 8. And the groups G(2p, p, 1) are cyclic of order 2, as is G(1, 1, 2).

11.4. Complexified alternating groups. Given a matrix α in a complexified symmetric group, there is a permutation matrix of the same size whose non-zero entries are in the same positions. We define the sign of the original matrix, $\operatorname{sgn}\alpha$, as the sign of this corresponding permutation matrix. We define $\operatorname{prod}\alpha$ to be the product of the non-zero elements, and thus $(\operatorname{prod}\alpha)^{k/p} = 1$. We also have $\det\alpha = \operatorname{sgn}\alpha \operatorname{prod}\alpha$.

This becomes interesting when we consider the subgroups of the complexified symmetric groups by requiring the corresponding permutation matrix to be in the alternating group, that is $\operatorname{sgn}\alpha = 1$. What new groups does this generate?

In the basic case, we define $A(n; k, r) = \{\alpha \in C(n; k, r) \mid \operatorname{sgn}\alpha = 1\}$.

The constraint on a complexified symmetric group is $(\operatorname{prod}\alpha)^{k/p} = 1$, suggesting that we also consider the constraint $(\det\alpha)^{k/p} = 1$.

Let's do this analysis: The generic constraints are $(\operatorname{prod}\alpha)^q = 1$, $(\det\alpha)^r = 1$, and $(\operatorname{sgn}\alpha)^s = 1$, where q and r divide k and s is either 1 or 2 (even when k is odd). What are the non-isomorphic groups that can be generated by chosing q, r, and s?

The constraint $(\det\alpha)^r = 1$ can be analyzed: $1 = (\det\alpha)^r = (\operatorname{prod}\alpha)^r(\operatorname{sgn}\alpha)^r$.

Thus the constraint

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\det\alpha)^r = 1 \text{ and } (\operatorname{sgn}\alpha)^s = 1$$

is equivalent to the constraint

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\operatorname{prod}\alpha)^r(\operatorname{sgn}\alpha)^r = 1 \text{ and } (\operatorname{sgn}\alpha)^s = 1$$

Case 1: $s = 1$: Then $\operatorname{sgn}\alpha = 1$. So the constraint is equivalent to

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\operatorname{prod}\alpha)^r = 1 \text{ and } \operatorname{sgn}\alpha = 1$$

$$(\operatorname{prod}\alpha)^{\gcd(q,r)} = 1 \text{ and } \operatorname{sgn}\alpha = 1$$

This is the group $A(n; k, k/\gcd(q, r))$.

Case 2: $s = 2$: Then $\operatorname{sgn}\alpha$ is unconstrained. So the constraint is equivalent to

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\operatorname{prod}\alpha)^r(\operatorname{sgn}\alpha)^r = 1$$

Case 2.1: r is even: Then the constraint is equivalent to

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\operatorname{prod}\alpha)^r = 1$$

$$(\operatorname{prod}\alpha)^{\gcd(q,r)} = 1$$

This is the group $C(n; k, k/\gcd(q, r))$.

Case 2.2: r is odd: Then the constraint is equivalent to

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\operatorname{prod}\alpha)^r \operatorname{sgn}\alpha = 1$$

This group is the union of two sets of elements based on whether $\operatorname{sgn}\alpha$ is 1 or -1.

Case 2.2.1: Elements with $\operatorname{sgn}\alpha = 1$. For these elements, the constraint is equivalent to

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\operatorname{prod}\alpha)^r = 1 \text{ and } \operatorname{sgn}\alpha = 1$$

$$(\operatorname{prod}\alpha)^{\gcd(q,r)} = 1 \text{ and } \operatorname{sgn}\alpha = 1$$

These elements are the group $A(n; k, k/\gcd(q, r))$.

Case 2.2.2: Elements with $\operatorname{sgn}\alpha = -1$. For these elements, the constraint is equivalent to

$$(\operatorname{prod}\alpha)^q = 1 \text{ and } (\operatorname{prod}\alpha)^r = -1 \text{ and } \operatorname{sgn}\alpha = -1$$

We know that $\gcd(q, r)$ is an integral linear comination of q and r, so $(\operatorname{prod}\alpha)^{\gcd(q,r)}$ is ± 1. But since $\gcd(q, r) \mid r$, we must have $(\operatorname{prod}\alpha)^{\gcd(q,r)} = -1$. Thus q is an even multiple of $\gcd(q, r)$ and since r is known to be odd, that is equivalent to q being even. Thus for this class of element to exist, q must be even, and then these elements satisfy the condition

$$(\operatorname{prod}\alpha)^{\gcd(q,r)} = -1 \text{ and } \operatorname{sgn}\alpha = -1$$

*** Likely something is going wrong here.

Define $\omega = \exp(\pi i / \gcd(q,r))$ so $\omega^{\gcd(q,r)} = -1$ and

$$J = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \omega & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n-2} \end{pmatrix} \qquad J^{-1} = \begin{pmatrix} 0 & \omega^{-1} & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n-2} \end{pmatrix}$$

and $\beta = J\alpha$. Since $(\operatorname{prod} J)^{\gcd(q,r)} = \omega^{\gcd(q,r)} = -1$ and $\operatorname{sgn} J = -1$, the condition becomes equivalent to

$$(\operatorname{prod} \beta)^{\gcd(q,r)} = 1 \text{ and } \operatorname{sgn} \beta = 1$$

Thus the set of β is $A(n;k,k/\gcd(q,r))$ and the set of α is the elements of $A(n;k,k/\gcd(q,r))$ multiplied by J^{-1}.

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We can define $A(n;k,r)$ as the preimage of -1 under the homomorphism sgn from $C(n;k,r)$.

In the case where $s = 2$, q is even, and r is odd, Define $\omega = \exp(\pi ir/k)$ so Define $\omega = \exp(2\pi ir/2k)$ so $\omega^{k/r} = -1$ and gcd = k/r which equals the other r. In terms of the other r, $\omega^r = -1$ and $\omega = \exp(2\pi i/2r)$

$A(n;k,r) \cup J^{-1}A(n;k,r)$

prod alpha k/r sgn a = 1

Order of $C(n;k,r) = n!k^n/r$. Order of $A(n;k,r) = n!k^n/2r$ except $A(1;k,r) = k/r$.

$A(n;k,r) \cup J^{-1}A(n;k,r)$ Order is $n!k^n/r$. But of course, this isn't a subset of $C(n;k,r)$ because $\omega = \exp(2\pi ir/2k)$. It is a subset of $C(n;k,2r)$. There is a homomorphism of $C(n;k,2r)$, $(\operatorname{sgn} \bullet, (\operatorname{prod} \bullet)^r)$, into $C_2 \times C_2$. This group is the preimage of $\{(1,1),(-1,-1)\}$ of $C(n;k,2r)$ under this map. Until I analyze this better, call it the *quaternary group* $Q(n;k,r)$. What is its order? Is it different from C and A?

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The summary of all these cases is

	q	r	s	
Case 1			1	$A(n;k,\frac{k}{\gcd(q,r)})$
Case 2.1		even	2	$C(n;k,\frac{k}{\gcd(q,r)})$
Case 2.2.1 alone	odd	odd	2	$A(n;k,\frac{k}{\gcd(q,r)})$
Case 2.2.1 and 2.2.2	even	odd	2	$A(n;k,\frac{k}{\gcd(q,r)}) \cup J^{-1}A(n;k,\frac{k}{\gcd(q,r)})$

Case 1 can generate any $A(n;k,r)$ by choosing $q = r$ and $r = 2r$. Case 2.1 can generate any $A(n;k,r)$ by choosing $q = r$ and $r = 2r$.

11.5. Schur covers. Since we expect the projective representations of groups to appear in the general theory, we include among the groups of interest the Schur covers of the above groups. Note that some groups (e.g., the hyperoctahedral groups) have multiple, non-isomorphic Schur covers, and each Schur cover generates a disjoint set of projective representations.

We also need to verify how the projective representations of a tower of groups themselves form a tower and verify that they have the product property.

`https://groupprops.subwiki.org/wiki/Category:Survey_articles` `https://groupprops.subwiki.org/wiki/Schur_covering_group` `https://groupprops.subwiki.org/wiki/Schur_cover_of_alternating_group:A7` `https://groupprops.subwiki.org/wiki/Schur_cover_of_alternating_group:A6` `https://groupprops.subwiki.org/wiki/Projective_representation_theory_of_alternating_groups` `https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_binary_octahedral_group` `https://groupprops.subwiki.org/wiki/Linear_representation_theory_of_alternating_groups`

11.6. Restriction and Induction. Given a group G and a subgroup H of G, there is of course an injective homomorphism (embedding, monomorophism) from H to G. This allows any representation ϕ_G of G to be restricted to a representation ϕ_H of H, denoted $\phi_H = \operatorname{Res}^G_H \phi_G$. More complicatedly, any representation ϕ_H of H can be used to construct the "induced" representation ϕ_G of G, denoted $\phi_G = \operatorname{Ind}^G_H \phi_H$. These are not inverse operations. Their behavior on irreps of G and H are given by Frobenius reciprocity.

11.7. Homomorphic images and representations. As a sort of dual, given a group G and a homomorphic image (quotient) H of G, there is a surjective homomorphism (epimorophism) q from G to H. Any representation ϕ_H can be used to construct a representation ϕ_G of G with $\phi_G(g) = q \circ \phi_H(h)$. Necessarily, ϕ_H is irreducible iff ϕ_G is irreducible.

Conversely, if a representation ϕ_G of G is constant on each coset of H, then ϕ_G can be "factored through" q to give a representation ϕ_H of H. This restriction works iff ϕ_G can be generated from a representation of H by the above extension process.

More interestingly, given a general ϕ_G, we can define

$$\phi_H(h) = [G:H]^{-1} \sum_{g \mid q(g)=h} \phi_G(g)$$

ϕ_G is multiplicative:

$$\begin{aligned}
\phi_H(h_1)\phi_H(h_1)[G:H]^2 &= \sum_{g_1 \mid q(g_1)=h_1} \phi_G(g_1) \sum_{g_2 \mid q(g_2)=h_2} \phi_G(g_2) \\
&= \sum_{\substack{g_1 \mid q(g_1)=h_1 \\ g_2 \mid q(g_2)=h_2}} \phi_G(g_1)\phi_G(g_2) \\
&= \sum_{\substack{g_1 \mid q(g_1)=h_1 \\ g_2 \mid q(g_2)=h_2}} \phi_G(g_1 g_2)
\end{aligned}$$

The set of g_1, g_2 pairs can be redefined as g, g_1 pairs, where $g = g_1 g_2$, $g_2 = g_1^{-1} g$, and the summation conditions are $q(g) = h_1 h_2$ and $q(g_2) = h_2$:

$$\begin{aligned}
&= \sum_{\substack{g \mid q(g)=h_1 h_2 \\ g \mid q(g_1)=h_1}} \phi_G(g) \\
&= [G:H] \sum_{g \mid q(g)=h_1 h_2} \phi_G(g) \\
&= [G:H]^2 \phi_H(h_1 h_2)
\end{aligned}$$

However that proves that ϕ_H is a representation only if $\phi_H(e_G)$ is invertible.

This looks like each irrep ϕ_G of G produces a ϕ_H which is an irrep if ϕ_G is constant on each coset of H and is zero otherwise. Does Schur's Lemma prove this?

(This analysis is based on an idea provided by Alex Postnikov.)

Call the quotient map $q : G \to H$. Given a representation ϕ_H of H you can easily construct a representation of G by "expansion": $\phi_G(g) = \phi_H(q(g))$.

The interesting question is whether there is a non-trivial way to "compress" a representation ϕ_G of G into a representation ϕ_H of H. Let me call the inverse image in G of each element of H a "co-set" for brevity. If the representations of all the elements in each co-set are the same, then setting $\phi_H(h) = \phi_G(g)$ for an arbitrary $g \in q^{-1}(h)$ gives a "compressed" representation of H.

One possible choice for a non-trivial "copression" is "averaging" the representations of the elements of each co-set: $\phi_H(h) = [G:H]^{-1} \sum_{g|q(g)=h} \phi_G(g)$. ϕ_H is multiplicative on its argument and seems to be workable. However, ϕ_H might be degenerate; it could be zero for some h or even map some subspace of the target vector space to zero for all h. This fact is the key to showing this "compression" isn't interesting.

Consider ϕ_G to be an irrep of G. "Averaging" is a linear map from ϕ_G to the "compressed" representation ϕ_H of H. Compose this with "expanding" ϕ_H into ϕ'_G. The composition is a linear map from the irrep ϕ_G to the representation ϕ'_G, and is onto by construction. By Schur's Lemma, the map is a constant times the identity map.

If the constant is zero: ϕ_H must be entirely zero, since its image in the representaton space GL(V) is the same as the image of ϕ'_G.

If the constant is nonzero: Since the value of ϕ'_G is constant on every co-set, ϕ_G must be constant on every co-set. Thus ϕ_G can be generated by expansion from some representation of H, specifically ϕ_H.

Thus, "averaging" of a general representation of G amounts to preserving each irrep summand of the representation that is constant on co-sets and zeroing all other irrep summands.

So the relationship of representations is that each irrep of H is associated with a distinct irrep of G, its "expansion". And "compressing" irreps of G compress any irrep that is constant on co-sets into the corresponding irrep of H, and compresses any other irrep to zero. The set of representations of H and the set of representations of G are related as subset and superset.

Clearly, this argument extends to any process for "compressing" a representation of G into a representation of H that is linear.

All of this explains why nobody writes about "the relationship of quotients and representations" – it's simple and produces no interesting splitting coefficients.

https://math.stackexchange.com/questions/5132817/representations-of-homomorphic-image-of-a-group/5133409

12. Generalized permutation groups

*** check into the terminology

https://en.wikipedia.org/wiki/Weyl_group

Groups of Weyl type (Still not clear if "Weyl type" is different from "Weyl group".)

Gemini: Weyl groups are finite reflection groups associated with Lie algebras, Lie groups, or root systems, generated by reflections through hyperplanes orthogonal to roots. They are finite Coxeter groups that classify semisimple Lie algebras by type (An,Bn,Cn,Dn,E6,E7,E8,F4,G2), with the Weyl group acting on the root system by permutation.

WikipediaWikipedia +2

Types of Weyl Groups

Type An (Ar): The Weyl group W(An) is the symmetric group Sn+1Sn+1 (or Sr+1Sr+1), which consists of all permutations of n+1 elements.

Type Bn (Br) and Cn (Cr): The Weyl groups W(Bn) and W(Cn) are isomorphic, often referred to as the signed permutation group or the hyperoctahedral group, acting on nn elements with sign changes.

Type Dn (Dr): The Weyl group W(Dn) is a subgroup of W(Bn) of index 2, consisting of signed permutations with an even number of sign changes.

Exceptional Types: These are finite groups associated with the exceptional Lie algebras:

E6,E7,E8: Large finite groups, with W(E8) being a large group related to O(8,F2)+.

F4: A solvable group of order 1152.

G2: The dihedral group of order 12.

Key Properties

Structure: Weyl groups are generated by reflections s_α corresponding to simple roots.

Quotient Definition: For a compact Lie group GG and a maximal torus T, the Weyl group is NG(T)/T, where NG(T) is the normalizer of TT.

Equivariant Homotopy: In this context, it can generalize to the quotient (NGH)/H for a subgroup HH.

Coxeter Group: Every Weyl group is a finite Coxeter group, which can be visualized by its Dynkin diagram.

These groups are crucial for understanding the symmetry of root systems and the representation theory of Lie algebras.

Affine Coxeter groups: Infinite but discrete reflection groups. https://en.wikipedia.org/wiki/Coxeter_group#Affine_Coxeter_groups

There are also hyperbolic Coxeter groups. https://en.wikipedia.org/wiki/Coxeter-Dynkin_diagram#Hyperbolic_Coxeter_groups

***

Sn wreath prod Ck has irreps indexed by k-tuples of partitions. So its lattice is Y to the k.

The "classical combinatorial groups" are a rather large set. We start with either the symmetric group or the alternating group. Then wreath product it with some $\mathbb{Z}_{\%k}$ to produce a generalized group. Then constrict that so the product of the entries is an r-th root of unity. These constructions produce all the S_n, B_n, and D_n groups.

From there we can take the Schur covering groups. This gets complicated because the Schur covering group of a group is not necessarily unique – for the hyperoctahedral group, there can be 6 different covering groups!

Google Gemini for "what is the schur cover of the hyperoctahedral group" gives: The hyperoctahedral group B_n (also denoted as $W(B_n)$ or $O_n(\mathbb{Z})$), which is the wreath product $(\mathbb{Z}/2\mathbb{Z})\wr S_n$, has seven distinct non-split double covers for $n \geq 4$. These double covers are associated with the projective representations of the gorup and are constructed using Schur's work on spin representations. The irreducible projective representations of the Weyl group of the root system D_n are also obtained from these covers.

Google Gemini for "what is the schur double cover of the symmetric group" gives: The Schur double cover of the symmetric grup S_n (for $n \geq 4$) is a non-split central extension of S_n by a cclic group of order 2 (C_2), often denoted as $2{\cdot}S_n$ or $S_n^\sim$, with order $2{\cdot}n!$. It is the representaiton group (Schur cover) that accounts for all projective representaitons of S_n, with two isomorphism classes for $n \neq 6$ and one for $n = 6$. See also `https://groupprops.subwiki.org`.

These can be assembled into towers in multiple ways due to the interaction of the k and r parameters.

The towers connect in multiple ways as well. The canonical connection is the subgroup relationship, which leads to Frobenius reciprocity. But there is the quotient relationship, whose effect on representations I do not know. (S_n is both a subgroup of B_n and a quotient of B_n.) There is also the Schur covering relationship, which is defining for projective representations.

13. Alternating groups

Google Gemini "representation thoery the alternating groups"

The representation theory of alternating groups (AnA) involves understanding how these groups of even permutations act on vector spaces, with complex irreducible representations closely related to those of the symmetric group S nS. For ≥ 5, AnA is simple, and its complex representations are derived from SnS representations by restricting characters, classifying irreducibles based on partition symmetry.

Wikipedia Wikipedia +4

Key Aspects of AnAn

Representation Theory

Relationship to SnSn: AnA is a normal subgroup of index two in SnS. Irreducible representations of SnS (S^λ) restricted to AnA either remain irreducible or split into two distinct, conjugate irreducible representations of AnA.

Irreducible Representations: The irreducible representations of AngAcorrespond to partitions λ of n. Specifically, if a partition λ is not equal to its conjugate partition λ', the representation S^λ restricted to AnA is irreducible. If $\lambda = \lambda'$ (a self-conjugate partition), S^λ splits into two irreducible representations of AnA with the same dimension.

Characters and Conjugacy Classes: The character table is derived from SnS by observing that conjugacy classes in SnS might split into two distinct classes in AnA (specifically classes of even permutations consisting of disjoint cycles of distinct odd lengths, including 1-cycles).Small Dimensions (n=5,6): A5A5 has irreducible representations of dimensions 1, 3, 3, 4, 5. A6A6 includes exceptional representations, such as those related to its exceptional automorphism, having two distinct 5-dimensional representations.

Projective Representations: The Schur multiplier of AnA is generally C2C2 (double cover), except for =6,7 where it is C6C6.

Groupprops Groupprops +4

Examples and Specific Cases

A3A3: Abelian (C3C3), three 1-dimensional representations.

A4A4: Has 4 irreducible representations: three 1-dimensional (via quotient $A4/V \cong C3$) and one 3-dimensional representation (action on a 3D subspace of a 4D space).

A5A5: The smallest non-abelian simple group, acting as rotational symmetries of an icosahedron.

Wikipedia Wikipedia +3

Linear representation theory of alternating groups - Groupprops

Groupprops `https://groupprops.subwiki.org/wiki/Linear_represent...` Aug 19, 2012 - This article discusses the linear representation theory of the alternating group A n of finite degree n .Read more (PDF) On the representation theory of the alternating groups

ResearchGate https://www.researchgate.net/publication/26830211... PDF — We present the basic results on the representation theory of the alternating groups Vn. Our approach is based on Clifford theory.

Representation theory of the symmetric group - Wikipedia The alternating groups for $n \geq 5$ have only one one-dimensional irreducible representation, the trivial representation. For n = 3, ...

Leibniz Universität Hannover https://www.iazd.uni-hannover.de/iazd/algcoll PDF The representation theory of the alternating groups has been studied from the early beginnings of the representation theory of finite groups. The starting ...Read more 10 pages

Google Gemini what is the schur multiplier of the alternating groups

The Schur multiplier of the alternating group AnAn (denoted $H2(An,\mathbb{Z})$) is generally the cyclic group of order 2, C2C2 (or $\mathbb{Z}/2\mathbb{Z}$), for $n \geq 4$. The main exceptions are A6A6 and A7A7, where the Schur multiplier is C6C6 (or $\mathbb{Z}/6\mathbb{Z}$) due to exceptional triple covers.

Schur Multiplier $H2(An,\mathbb{Z})$ by degree nn: $n \leq 3$: $H2(An,\mathbb{Z}) = 0$ (trivial). n=4,5: $H2(An,\mathbb{Z}) \cong C2$. n=6,7: $H2(An,\mathbb{Z}) \cong C6$. $n \geq 8$: $H2(An,\mathbb{Z}) \cong C2$.

Wikipedia Wikipedia

Key Details:

Double Covers: For $n \geq 4$ (excluding 6, 7), the Schur cover is a double cover, often denoted $2 \cdot An$. For example, the Schur cover of A4A4 is SL(2,3).

Triple Covers: A6A6 and A7A7 are unique in that their covering groups are 6-fold covers (often denoted $6 \cdot A6$ and $6 \cdot A7$).

Context: The Schur multiplier classifies the central extensions of AnAn, and it is used to understand projective representations of alternating groups.

`https://www.google.com/url?sa=t&source=web&rct=j&opi=89978449&url=https://en.wikipedia.org/wiki/Schur_multiplier&ved=2ahUKEwjV-ZKk7sCTAxUJFlkFHQrIDFkQFnoECBsQAQ&usg=AOvVaw3yW4cdpUugpYxmfTkmsZ9q` `https://www.google.com/url?sa=t&source=web&rct=j&opi=89978449&url=https://en.wikipedia.org/wiki/Covering_groups_of_the_alternating_and_symmetric_groups&ved=2ahUKEwjV-ZKk7sCTAxUJFlkFHQrIDFkQFnoECCIQAQ&usg=AOvVaw0NxS8azhOMYgaJFGkQ6CvR` `https://www.google.com/url?sa=t&source=web&rct=j&opi=89978449&url=https://groupprops.subwiki.org/wiki/Schur_multiplier&ved=2ahUKEwjV-ZKk7sCTAxUJFlkFHQrIDFkQFnoECB0QAQ&usg=AOvVaw0ZVf3I55Z3VcjOc53rvgx8` `https://www.google.com/url?sa=t&source=web&rct=j&opi=89978449&url=https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-10/issue-2/Review--F-Rudolf-Beyl-and-J%25C3%25BCrgen-Tappe-Group-extensions/bams/1183551591.pdf&ved=2ahUKEwjV-ZKk7sCTAxUJFlkFHQrIDFkQFnoECCMQAQ&usg=AOvVaw3Jg8IWG1CRpOm55GZ1OeOD`

Other complex reflection group families?

"Groups of Weyl type"

14. Schur covers and projective representations

https://iopscience.iop.org/article/10.1088/0305-4470/14/2/009 Journal of Physics A: Mathematical and General The symmetric group: branching rules, products and plethysms for spin representations L Dehuai and B G Wybourne

https://academic.oup.com/book/53389/chapter-abstract/422058168?redirectedFrom=fulltext 10 Reduction, Branching and Degree Formulae Get access Arrow P N Hoffman, J F Humphreys

This is a Google Gemini answer about Clifford theory:

Given a group G which is the extension of the quotient group H by the normal subgroup N, can the representations of G be constructed from the representations of H and the representations of N?

Yes—in principle, but not in a simple plug-and-play way. You can build representations of GG from those of NN and H=G/NH=G/N, but you need extra structure, and not every pair of representations combines nicely. The basic setup You have a short exact sequence:1→N→G→H→11→N→G→H→1 with NN normal in GG, and you want to understand representations of GG in terms of those of NN and HH. This is exactly the domain of Clifford theory.

Key idea: restrict, then extend Take an irreducible representation π of GG. When you restrict it to NN:π it typically decomposes into irreducible representations of NN. The crucial facts are: So GG acts on N^N , the set of irreducible representations of NN. The construction (big picture) To build representations of GG: 1. Start with an irreducible representation of NN Pick $\sigma \in N$. 2. Look at its stabilizer in GG Define the inertia subgroup:$G\sigma = g \in G : \sigma g \cong \sigma G\sigma = \{g \in G : \sigma g \cong \sigma\}$ 3. Try to extend $\sigma\sigma to G\sigma\ G\sigma$ This is the first obstruction: Not every $\sigma\sigma$ extends to a representation of $G\sigma G\sigma$. Sometimes you only get a projective representation, tied to a cohomology class. This obstruction lives in Group cohomology. 4. Combine with representations of the quotient If σ extends (or can be modified appropriately), then: You tensor it with a representation of $G\sigma/N \subset HG\sigma/N \subset H$, Then induce up to GG. This uses Induced representation. When things are nice Case 1: Direct product $G = N \times HG = N \times H$ This is the cleanest situation: Every irreducible representation is just:$\sigma \otimes \tau$ where $\sigma \in N^x\sigma \in N, \tau \in H^x\tau \in H$.

Case 2: Semidirect product $G = N semidirect H$ Still manageable:HH acts on N^N, Representations are built via: orbits of this action, stabilizers, induction. This is often called the "little group method" (common in physics too). Case 3: General extension Now complications appear: Extensions may not split, Representations of NN may fail to extend, You may need projective representations of HH instead of honest ones. Bottom line Yes, representations of GG are systematically built from those of NN plus data fromHH. But: You must account for how HH acts on representations of NN, You may face extension obstructions, And you often need induction and cohomology to complete the construction. If you want, I can walk through a concrete example (like the dihedral group or S3) where this construction is fully explicit.

15. Representation of modular lattices

I know of three attacks on this: DavPriest1990a ch. 10. There is a paper by Herrmann with a th. that if any ji is in the closure of a set of ji's it is in the closure of two of the ji's in the set. factors-atoms.tex attacks based on dissection/gluing. Ther may be stuff in notes-* files.

Also see BensConw1985a, but it applies only to finite length.

Also see Day1982a.

From NYCD 2026: Speaker: Kira Adaricheva (Hofstra University) Title: Closure Systems with unique critical sets Abstract: Closure systems are ubiquitous in mathematics and they appear in two major representations: as families of closed set or via implicational bases. In the first form they are lattices, while in the second the implications represent the behavior of the underlying closure operator. The finite join-semidistributive lattices are hard to recognize given their implicational basis. This led to the definition of the class of closure systems with unique critical sets (UC-systems) - naturally defined and recognized via their canonical basis. UC-systems include all (finite) join semidistributive closure systems, as well as their important subclasses: systems without cycles and convex geometries. We will introduce UC-systems and discuss several aspects of them, both structural and algorithmic, as well as offer several open problems. This is the work in progress in collaboration with S. Vilmin (University of Aix-Marseille, CNRS, LIS, France)

16. Representation of general lattices

DavPriest1990a ch. 8 represents general lattice that are finite.

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