Extending Birkhoff's representation theorem to modular lattices

The Many Combinatorial Legacies of Richard P. Stanley
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Goal: Construct a nice way to represent/classify modular lattices

Applications:
☞ Construct "tableaux" that nicely represent saturated chains in general modular lattices.
☞ Find/classify (weighted) differential structures on general modular lattices for use in RSK algorithms.

1. Extending Birkhoff's representation theorem to modular lattices

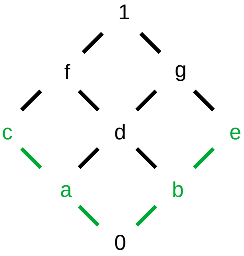
The goal of this research is to find a nice representation of modular lattices, as close as possible to Birkhoff's representation of distributive lattices.

Representing elements of a modular lattice as suitably constrained subsets of some base set allows saturated chains in the modular lattice to be represented as tableaux by assigning sequential integers to elements of the base set under suitable` constraints.

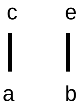
Birkhoff represents the elements of the lattice of partitions as ideals of the base set of join-irreducible elements of that lattice, which are the rectangular partitions, which can be indexed by pairs of positive integers. This representation allows saturated chains in the lattice of partitions to be represented as tableaux by assigning sequential integers to elements of the base set, the pairs of positive integers.

Birkhoff's representation of distributive lattices [Birkhoff, 1935]

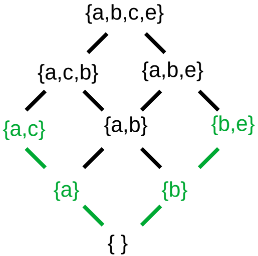
distributive lattice



poset of join-irreducible elements



ideals of the poset



join irreducibles
shown in green

- meet becomes intersection:
  $a \wedge e = 0 \Rightarrow \{a\} \cap \{b,e\} = \{\ \}$
- join becomes union:
  $a \vee e = g \Rightarrow \{a\} \cup \{b,e\} = \{a, b, e\}$

2. Birkhoff's representation of distributive lattices

Birkhoff's representation theorem is a beautiful, classical result.

The representation is constructed by first identifying the join-irreducible elements of the lattice, those that cannot be expressed as a join of smaller elements. The join-irreducible elements are the elements that cover exactly one element.

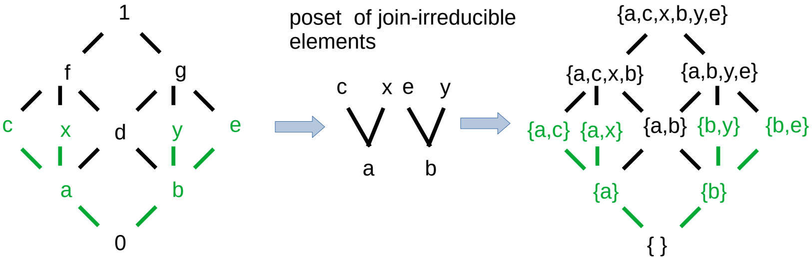
On the left, the lattice has four join-irreducible elements.

The join-irreducibles inherit a partial ordering from the lattice, as shown in the center.

On the right is the lattice of ideals of the poset, showing that the original lattice is isomorphic to the lattice of ideals of the poset of join-irreducibles.

Any saturated chain in this lattice can be represented by numbering a subset of the join-irreducible elements compatibly with their poset ordering.

The trouble with non-distributive modular lattices


modular lattice

poset of join-irreducible elements

closed ideals of the poset

- ☞ meet remains intersection:
  a ∧ e = 0 ⇒ {a} ∩ {b,e} = { }
- ☞ join becomes closure of union:
  a ∨ e = g ⇒ {a} ∪ {b,e} =
  cl({a, b, e}) = {a, b,y, e}

☞ the task is to characterize cl(·)

3. The trouble with non-distributive modular lattices

The difficulty with non-distributive lattices is that to recover the original lattice from the poset of join-irreducibles, we must take only closed ideals of the poset, for some particular closure operator. The closure operator captures the non-distributivity of the lattice.

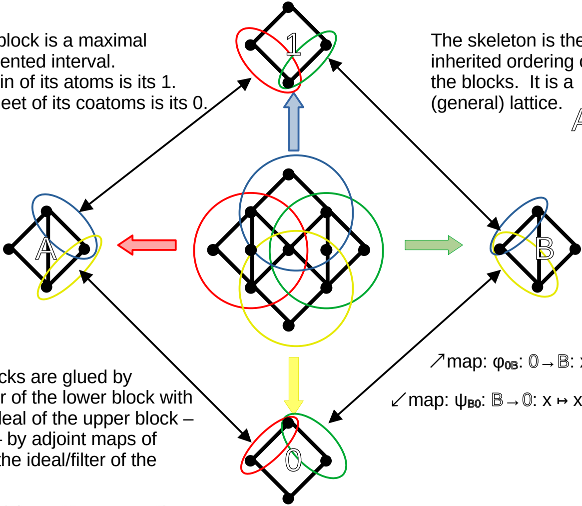
The meet of elements remains represented by the intersection of ideals just as in the distributive case. But the join becomes the closure of the union of ideals rather than just the union.

The fundamental task is to characterize these closure operators.

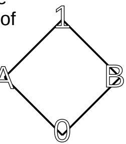
The closure operator is characterized by which join-irreducibles it adds to any given set of join-irreducibles. This can be reduced to characterizing the minimal sets which cause closure to add each particular join-irreducible to the set.

Dissecting a lattice into blocks [Herrmann, 1973]

☞ Each block is a maximal complemented interval.
☞ The join of its atoms is its 1.
☞ The meet of its coatoms is its 0.



The skeleton is the inherited ordering of the blocks. It is a (general) lattice.



Comparable blocks are glued by identifying a filter of the lower block with an isomorphic ideal of the upper block – or equivalently – by adjoint maps of each block into the ideal/filter of the other block.

↗map: $\varphi_{0B}: 0 \to B: x \mapsto x \vee 0_B$

↙map: $\psi_{B0}: B \to 0: x \mapsto x \wedge 1_0$

4. Dissecting a lattice into blocks

Herrmann introduced dissecting a lattice into overlapping blocks, where the blocks are the maximal complemented intervals. Here, the lattice in the center has its four blocks shown around it.

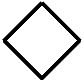
The set of blocks is called the skeleton of the lattice. The skeleton inherits a lattice ordering from the lattice. The skeleton is shown in the upper-right.

The lattice is then characterized by the blocks, the skeleton that orders the blocks, and exactly how each block overlaps with its neighbor blocks, called gluing.

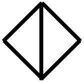
Two blocks are glued by identifying a filter of the lower block with an isomorphic ideal of the upper block. This identification generates a pair of adjoint maps between the two blocks. Typical adjoint maps are shown in the lower-right.

Modular blocks are products of projective geometries [Birkhoff, 1967]

☞ Blocks of modular lattices inherit modularity from the overall lattice.
☞ Complemented modular lattices are uniquely products of projective geometries, if you define projective geometries correctly.
☞ Thus, the building blocks of modular lattices are the projective geometries, which includes the non-Arguesian projective planes (ugh).
☞ The atoms of blocks are grouped according to which factor they are atoms of.

$\mathbb{0} \cong \mathbb{1} \cong$  $\cong$ | × | $\cong$ PG(0) × PG(0)

☞ the product of two “projective points”

$\mathbb{A} \cong \mathbb{B} \cong$  $\cong$ PG(1, 2)

☞ the projective line with 3 points (which is over F_2).

☞ For distributive lattices, all factors are PG(0), so all blocks are Boolean lattices.

5. Modular blocks are products of projective geometries

For modular lattices, the blocks are modular and complemented. This means that the blocks are the products of projective geometries.

Unfortunately, the incidence lattices of non-Arguesian projective planes are possible factors of blocks, which means that any full characterization of modular lattices depends on the full characterization of projective planes.

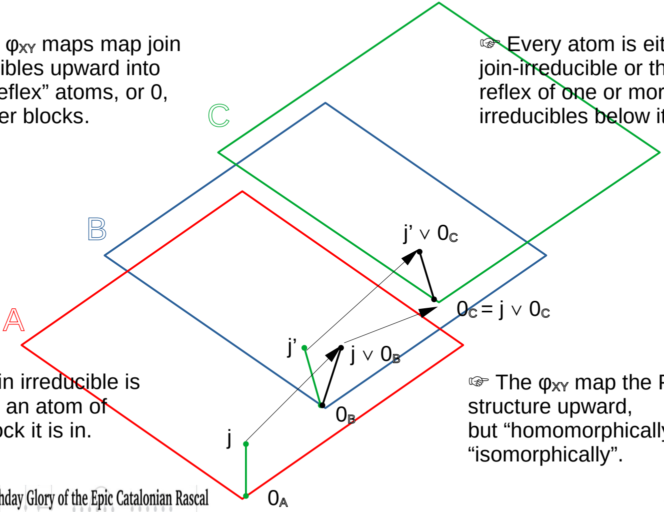
Here we see that two blocks of the example are isomorphic to the Boolean algebra with two atoms, which is the product of two projective points. The other two blocks are projective lines with three points each.

And here we see that the non-distributivity of modular lattices comes from the existence of factors that aren't projective points.

The relationships of the atoms of the blocks

☞ The φ_{XY} maps map join irreducibles upward into their “reflex” atoms, or 0, in higher blocks.

☞ Every atom is either join-irreducible or the reflex of one or more join-irreducibles below it.



☞ A join irreducible is always an atom of any block it is in.

☞ The φ_{XY} map the P.G. structure upward, but “homomorphically”, not “isomorphically”.

6. Relationships of the atoms of the blocks

Characterizing the closure operator of the representation of a modular lattice requires determining which join-irreducible elements are less than or equal to the join of a given set of join-irreducibles.

This test can be done entirely within one block containing the join. The test result is unchanged by replacing all of the join-irreducibles involved by their joins with the zero of the block.

The join of a join-irreducible atom of one block with the zero of a higher block is either the zero of the higher block or an atom of the higher block. This join is called the reflex of the atom in that block.

Thus the closure operator is characterized by two things: One is the closure of sets of atoms of within the blocks, which is determined by the projective geometry structure of the blocks. The other is the pattern of how join-irreducibles are mapped into their reflexes in higher blocks.