Representation of locally-finite distributive lattices

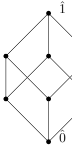
Dale R. Worley
https://alum.mit.edu/www/worley/Math

Brandeis Combinatorics Seminar

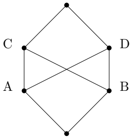
13 April 2026

Introduction to lattices

A lattice is a partially ordered set in which every pair of elements a and b has a "join" $a \vee b$ – least upper bound – and a "meet" $a \wedge b$ – greatest lower bound.



A lattice



A poset that is not a lattice

If a lattice has a minimum element, it is called $\hat{0}$. If a lattice has a maximum element, it is called $\hat{1}$.

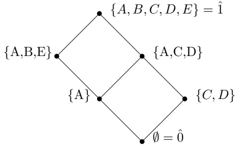
Distributive lattice

A distributive lattice is one for which two equivalent distributive laws hold:

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

As you might expect, this is a strong constraint on a lattice.

The standard examples of distributive lattices are collections of sets ordered by containment, where the meet is set intersection and the join is set union.

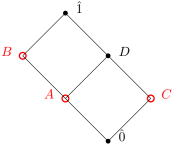


Join-irreducible elements

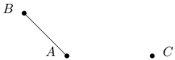
An element is *join-reducible* if it can be written as the join of two smaller elements or if it is $\hat{0}$ (which is the join of zero elements). An element is *join-irreducible* if it is not join-reducible.

If a lattice is locally-finite, then an element is join-irreducible iff it covers exactly one element.

The set of join-irreducible elements of a lattice forms a poset, with elements inheriting their relative orders in the lattice.



Lattice

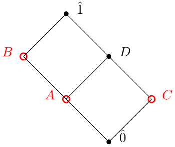


The join-irreducible elements

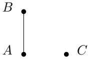
Birkhoff's Representation Theorem

Theorem (Birkhoff 1937)

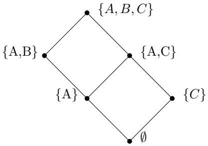
A finite distributive lattice is isomorphic to the lattice of ideals of the poset of join-irreducible elements of the lattice.



Lattice



The join-irreducible elements



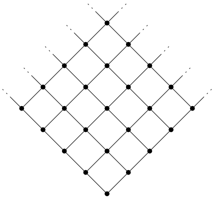
Lattice, reconstructed from the ideals of join-irreducible elements

Finitary lattices

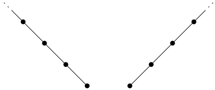
A *principal ideal* of a poset/lattice is the set of elements $\leq$ a given element.

A lattice is *finitary* if every principal ideal is finite.

Thus a finitary lattice is “finite downward” but may be “infinite upward” or “infinite sideways”.



The lattice $\mathbb{N} \times \mathbb{N}$

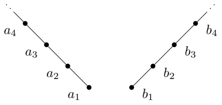


The join-irreducible elements

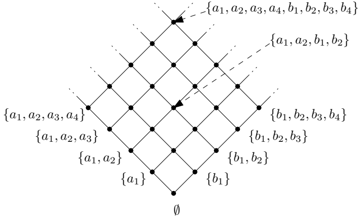
Representation of finitary lattices

Theorem (Stanley 2012, prop. 3.4.3)

A finitary *distributive lattice is isomorphic to the lattice of* finite *ideals of the poset of join-irreducible elements of the lattice.*



The join-irreducible elements of $\mathbb{N} \times \mathbb{N}$

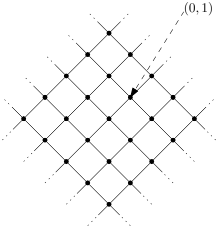


Lattice of ideals of the join-irreducible elements

Representation of general distributive lattices

(Stone 1938; Davey & Priestly 1990, ch. 10)

A central problem is that a general distributive lattice may have *no* join-irreducible elements. Consider $\mathbb{Z} \times \mathbb{Z}$:



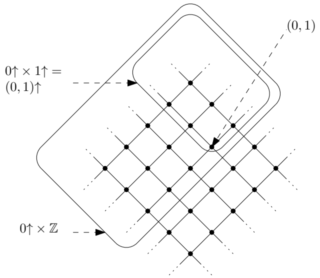
So what do we use to construct a representation instead of the join-irreducible elements?

Lattice filters

We consider *lattice filters*: A filter F is a non-empty subset of the lattice L for which:

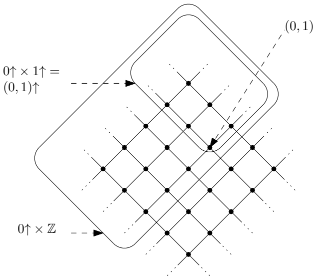
- If $x \in F$ and $y \in L$ with $x \leq y$, then $y \in F$.
- If $x, y \in F$, then $x \wedge y \in F$.

For any element $x \in L$, there is a *principal filter* $x\uparrow$, the set of all elements $\geq x$. But there can be non-principal filters as well:

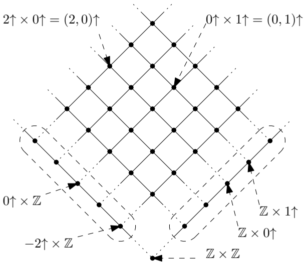


The lattice of lattice filters

The filters form a lattice under *inverse containment*, with the join being set intersection and the meet being meet-closure-of-set-union.



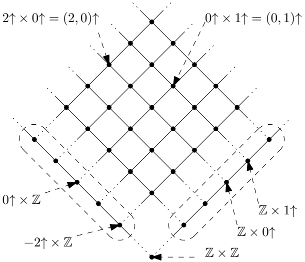
The lattice $\mathbb{Z} \times \mathbb{Z}$ and some of its filters



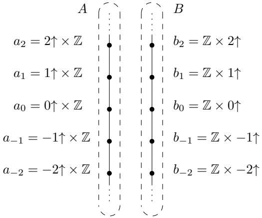
The lattice of filters of $\mathbb{Z} \times \mathbb{Z}$

The poset of prime filters

A join-irreducible element within the lattice of filters is called *prime*. Equivalently, a filter is prime iff the filter is not the whole lattice and whenever $x \vee y$ is in the filter, either x or y is. A principal filter is prime iff its generating element is join-irreducible. But there can be additional prime filters.



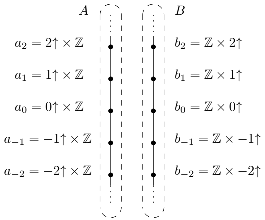
The lattice of filters of $\mathbb{Z} \times \mathbb{Z}$



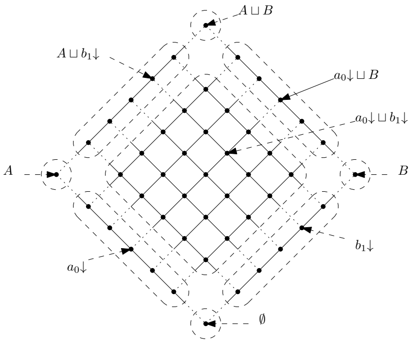
The poset of prime filters of $\mathbb{Z} \times \mathbb{Z}$, none of which are principal

Lattice of ideals of prime filters

As the final step, we construct the lattice of ideals of the poset of prime filters of the original lattice.(!) The original lattice is a sublattice of this lattice of ideals.



The poset of prime filters of $\mathbb{Z} \times \mathbb{Z}$



The lattice of ideals of prime filters

The general theorem

Theorem (Stone 1938; Davey & Priestly 1990)

A distributive lattice is isomorphic to a sublattice of the lattice of ideals of the poset of prime filters of the lattice.

Thus, every distributive lattice is isomorphic to a lattice of sets, where join is set union and meet is set intersection.

Locally-finite distributive lattices

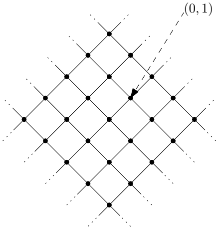
Given two elements $x \leq y$ in a lattice (or poset), the *interval* $[x, y]$ is the set of elements z for which $x \leq z \leq y$.

A lattice is *locally-finite* if every interval $[x, y]$ contains only a finite number of elements.

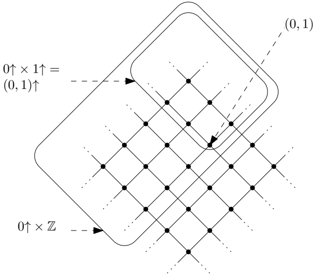
Of course, a locally-finite lattice may be infinite, but only "infinite outward". Many useful lattices in combinatorics are locally-finite but not finite or even finitary. Our example of $\mathbb{Z} \times \mathbb{Z}$ is one such.

In particular, you can get from any one element of the lattice to any other by walking up and down a finite number of cover relationships between elements.

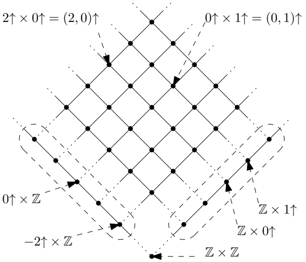
The lattice $\mathbb{Z} \times \mathbb{Z}$



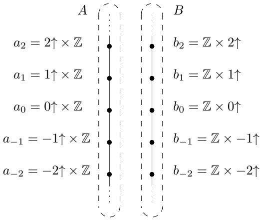
The lattice $\mathbb{Z} \times \mathbb{Z}$



Some filters

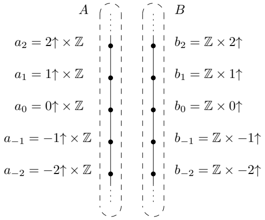


The lattice of filters

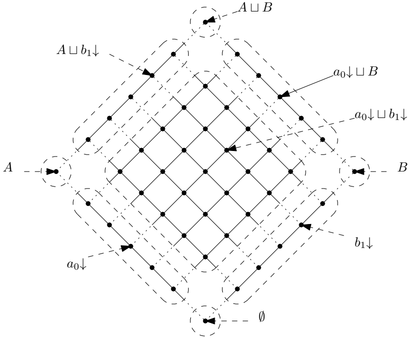


The poset of prime filters

Connected components



The poset of prime filters



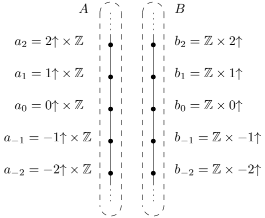
The lattice of ideals of prime filters

Looking at the lattice of ideals as a graph, it has 9 connected components. The central component is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

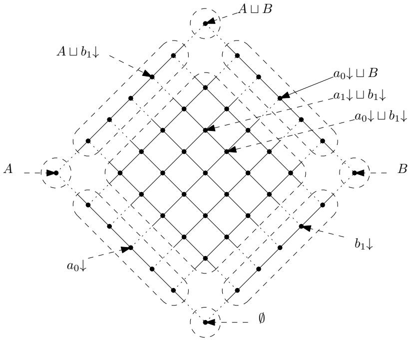
Single steps between ideals 1

With a locally-finite lattice:

- If one ideal covers another, they differ by exactly one prime filter.



The poset of prime filters

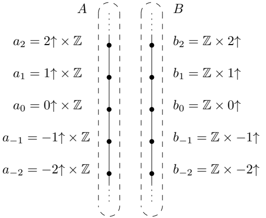


The lattice of ideals of prime filters

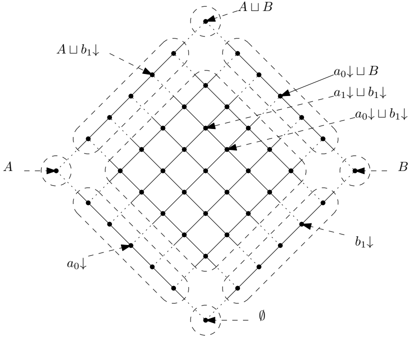
$a_1\downarrow \sqcup b_1\downarrow$ (corresponding to $(1,1)$) covers $a_0\downarrow \sqcup b_1\downarrow$ (corresponding to $(0,1)$), and includes exactly one more ideal, a_1.

Single steps between ideals 2

- If an ideal corresponds to an element of the lattice and you can add a prime filter to the ideal and make another ideal, the resulting ideal corresponds to an element of the lattice.



The poset of prime filters

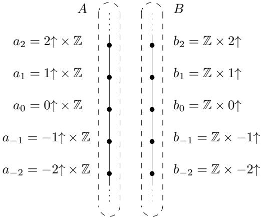


The lattice of ideals of prime filters

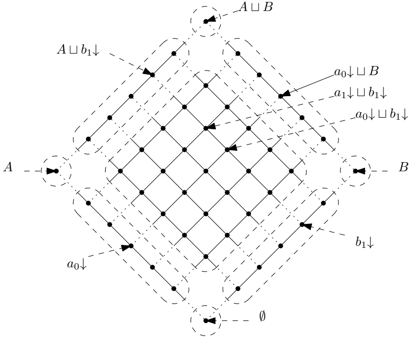
a_1 can be added to $a_0\downarrow \sqcup b_1\downarrow$ (corresponding to $(0, 1)$), giving $a_1\downarrow \sqcup b_1\downarrow$ (corresponding to $(1, 1)$).

Single steps between ideals 3

- Dually, if an ideal corresponds to an element of the lattice and you can delete a prime filter from the ideal to make another ideal, the resulting ideal corresponds to an element of the lattice.



The poset of prime filters



The lattice of ideals of prime filters

a_1 can be deleted from $a_1\downarrow \sqcup b_1\downarrow$ (corresponding to $(1, 1)$), giving $a_0\downarrow \sqcup b_1\downarrow$ (corresponding to $(0, 1)$).

Representation of locally-finite lattices

Theorem (W 2026+)

A locally-finite distributive lattice is isomorphic to one connected component of the lattice of ideals of the poset of prime filters of the lattice. The component consists of all ideals whose set symmetric difference from some particular fixed ideal is finite.

Questions?

References

Garrett Birkhoff, *Rings of sets*, Duke Math. J. 3 (1937), 443–454.

Brian A. Davey and Hilary Ann Priestley, *Introduction to Lattices and Order*, Cambridge Univ. Press, Cambridge, UK, 1990.

Richard P. Stanley, *Enumerative Combinatorics*, Vol. 1, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, 2012.

Marshall Harvey Stone, *Topological representations of distributive lattices and Brouwerian logics*, Čas. Mat. Fys. 67 (1938), 1–25.

Dale R. Worley, An extension of Birkhoff's representation theorem to locally-finite distributive lattices (2026), arXiv 2603.05841.