CLASSIFICATION OF LATTICES UNDERLYING SCHENSTED CORRESPONDENCES

DALE R. WORLEY

Abstract. The celebrated Robinson–Schensted algorithm and each of its variants that have attracted substantial attention can be constructed using Fomin's "growth diagram" construction from a modular lattice that is also a weighted-differential poset. In this article we classify all such "Fomin" lattices that meet certain criteria; the main criterion is that the lattice is distributive. Intuitively, these criteria seem excessively strict, but all known Fomin lattices satisfy all of these criteria, with the sole exception of the one family that is not even distributive, the Young–Fibonacci lattices and cartesian products involving them. This is the first classification theorem for weighted-differential Fomin lattices.

1. Introduction

The celebrated Robinson–Schensted algorithm[Rob1938][Schen1961][Fom1994][Fom1995a] and each of its variants that have attracted substantial attention can be constructed[Wor2023c, sec. 2] using Fomin's "growth diagram" construction from a modular lattice that is also a (weighted-)differential poset[Stan1988]. In this article we classify all Fomin lattices that meet certain criteria. Our main criterion is that the lattice is distributive, with secondary criteria on the set of points (the join-irreducible elements of the lattice). Intuitively, these criteria seem excessively strict, but all known Fomin lattices satisfy all of these criteria, with the sole exception of one family that is not even distributive, the Young–Fibonacci lattices (and cartesian products involving them). This work adds to the classes of lattices within which all of the Fomin lattices have been classified.

All known Fomin lattices satisfying these criteria are described in sec. 2.2. They are Young's lattice of partitions, the shifted Young's lattice of strict partitions, and the lattices of partitions into a limited number of parts.

The previous classification theorems for Fomin lattices are discussed in sec. 2.3.

Our new classification theorem is in sec. 6. The class of lattices classified by our theorem is incomparable with the classes of previous classification theorems; it does not supersede them. However, it is an incremental contribution to the classification of Fomin lattices (and thus Robinson–Schensted algorithms) and is the only one that applies to "weighted-differential" lattices, that is, where the weight on coverings (or points) is not identically 1.

Our proof techniques are elementary. They consist of a careful accounting of the "local" consequences of the differential condition (1) in lattices that satisfy our criteria. These results culminate in lem. 4.26 and 4.27, that no point covers three distinct points or is covered by three distinct points. We then proceed to the "global characterization" (sec. 5) of how the points can be placed on a two-dimensional grid. This culminates in the classification theorem, th. 6.1.

Finally, we discuss possible future directions for research (sec. 7), which assesses the prospect of various relaxations of the criteria of the classification theorem.

An expanded preprint of this article is available as [Wor2026b].

2. Scope of the classification

We call the lattice under consideration L, its weighting w, and the differential degree r.

2.1. Criteria.

We will classify lattices which satisfy these criteria:

(A) The lattice L is a *Fomin lattice*,[1] which comprises:

  (1) it is modular,

Date: January 29, 2026.
2020 *Mathematics Subject Classification.* Primary 06A11; Secondary 05A17, 06B99.
Key words and phrases. differential poset, Fomin lattice, graded graph, growth diagram, Robinson–Schensted algorithm, Young diagram, Young tableaux.

[1] We call it a Fomin lattice because all of the required properties are explicitly or implicitly given in [Fom1994] and [Fom1995a].

(2) it is locally finite,
(3) it has finite upward and downward covers for each element,
(4) there are one or more pairs of a *differential degree* r and a *weighting* $w(\bullet \lessdot \bullet)$, which is a valuation on *coverings* (or *prime quotients*), pairs $x \lessdot y$ of elements of L, for which L is a *weighted-differential lattice*[Stan1990, sec. 3][Fom1994, sec 2.2]:

$$\sum_{y \mid y \lessdot x} w(y \lessdot x) + r = \sum_{z \mid x \lessdot z} w(x \lessdot z) \qquad \text{for all } x \in L \tag{1}$$

(which is often described as r-*differential*); and
(5) for which w is projective-constant, that is, constant on projectivity classes[Birk1967, §I.7 Def.] of coverings, or equivalently, on coverings which are *cover-projective*: if $x \vee y$ covers x and y and $x \wedge y$ is covered by x and y, then $w(x \wedge y \lessdot x) = w(y \lessdot x \vee y)$ and $w(x \wedge y \lessdot y) = w(x \lessdot x \vee y)$.

(B) The lattice L has a minimum element, $\hat{0}_L$.
(C) The lattice L is distributive. We define $P = J(L)$ to be the poset of its join-irreducible elements, which we call *points*.
(D) The lattice L cannot be factored into the cartesian product of two non-trivial lattices, which is equivalent to that P cannot be partitioned into the disjoint union of two non-empty posets. This does not restrict the effective scope of our classification because we show in sec. 3 that if an L that meets the other criteria can be factored, both factors must themselves meet the other criteria. Conversely, the cartesian product of two lattices that meet the other criteria itself meets the other criteria.
(E) The poset of points P is *unique-cover-modular*.

Definition 2.1. *A poset P is* unique-cover-modular[2] *if: when $x, y \in P$ and either x and y both cover an element of P or x and y are covered by an element of P, then they both cover a unique element of P and they are both covered by a unique element of P.*

(F) The weighting and differential degree are *positive*. By abuse of language, we will call the weighting alone positive, or even call the lattice positive.

Definition 2.2. *A weighting w and differential degree r (taken together) are* positive *if all of the values of w are positive, and r is positive.*

This set of criteria may seem to be an excessively narrow approach, but the only known un-factorable Fomin lattices with $\hat{0}$ that do not satisfy all of these criteria are $\mathbb{YF}_\bullet$, the Young–Fibonacci lattices.

2.2. Known cases.

All known Fomin lattices that meet our criteria can be constructed as follows:

(1) [Fom1994, Exam. 2.1.2 and 2.2.7] Young's lattice $\mathbb{Y}$, the lattice of partitions, with $r = 1$ and $w = 1$. See fig. 1. (There are additional weightings w on this lattice that make it Fomin but are not positive.)
(2) [Sag1979b][Wor1984][Fom1994, Exam. 2.2.8] The shifted Young's Lattice $\mathbb{SY}$, the lattice of partitions into unequal parts, with $r = 1$ and $w = 1$ or 2. See fig. 2.
(3) [Fom1994, Exam. 2.2.5] For any integer $k \geq 1$, the k-row Young's lattice, $\mathbb{Y}_k$, the lattice of partitions into at most k parts (or dually, into an arbitrary number of parts that are all $\leq k$), with $r = k$ and w having a particular range of values. See $\mathbb{Y}_3$ in fig. 3.
(4) [Fom1994, Exam. 2.2.1] The upward semi-infinite chain $\mathbb{N}$ with $r = k$ and $w(i) = k(i+1)$ (which is isomorphic to $\mathbb{Y}_1$, a special case of (3)).
(5) Given a known Fomin lattice that meets these criteria, the same lattice but multiplying the differential degree and weighting by the same positive constant.

2.3. Previous classifications.

The previous classifications of Fomin lattices with $\hat{0}$ are:

(1) [Stan1988, Prop. 5.5] If a Fomin lattice with $\hat{0}$ is distributive and has $w = 1$, then it is isomorphic to $\mathbb{Y}^{\times r}$.[3]
(2) [Byrn2012, Th. 1.3] If a Fomin lattice with $\hat{0}$ has $r = 1$ and $w = 1$, then it is either $\mathbb{Y}$ or $\mathbb{YF}_1$.

In particular, there are no previous classifications for the weighted case when w is not identically 1.

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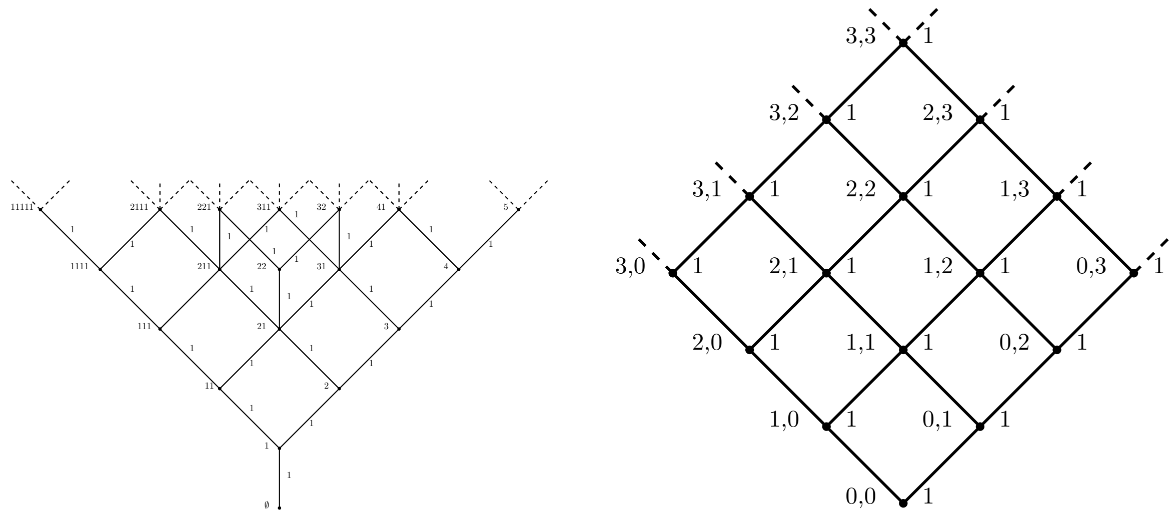
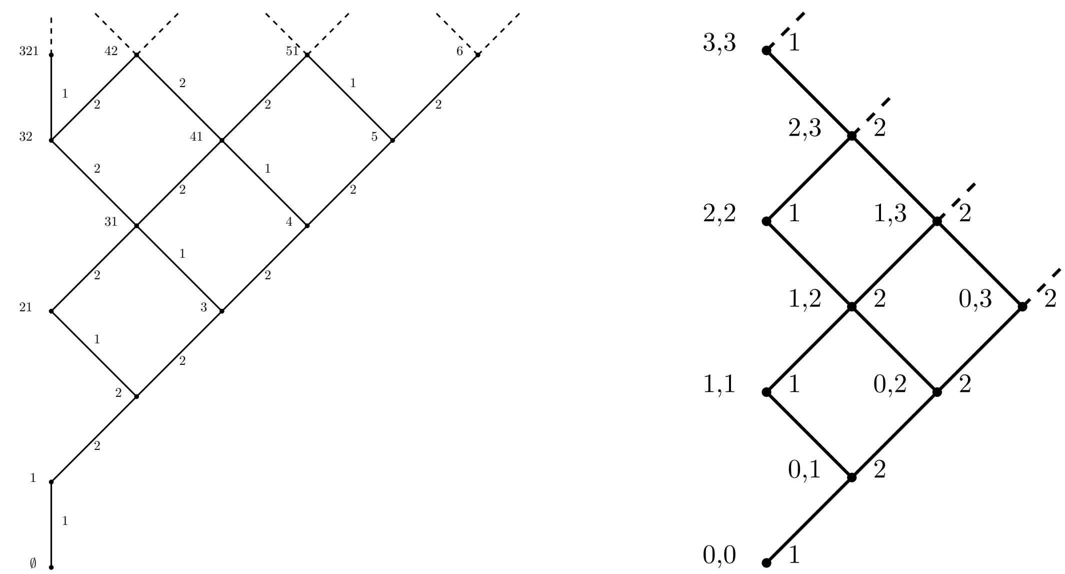
[2] This property is called a "uniquely modular poset" in [Proct1982, sec. 5].
[3] We use $\bullet^{\times n}$ to denote the n-fold cartesian exponential to avoid ambiguity with other uses of superscripts.



(A) Lattice $\mathbb{Y}$ with weights of the coverings (B) The quadrant $J(\mathbb{Y})$ with weights of the points

Figure 1. Young's lattice $\mathbb{Y}$, the lattice of partitions, with its canonical weighting.



(A) Lattice $\mathbb{SY}$ with weights of the coverings (B) The octant $J(\mathbb{SY})$ with weights of the points

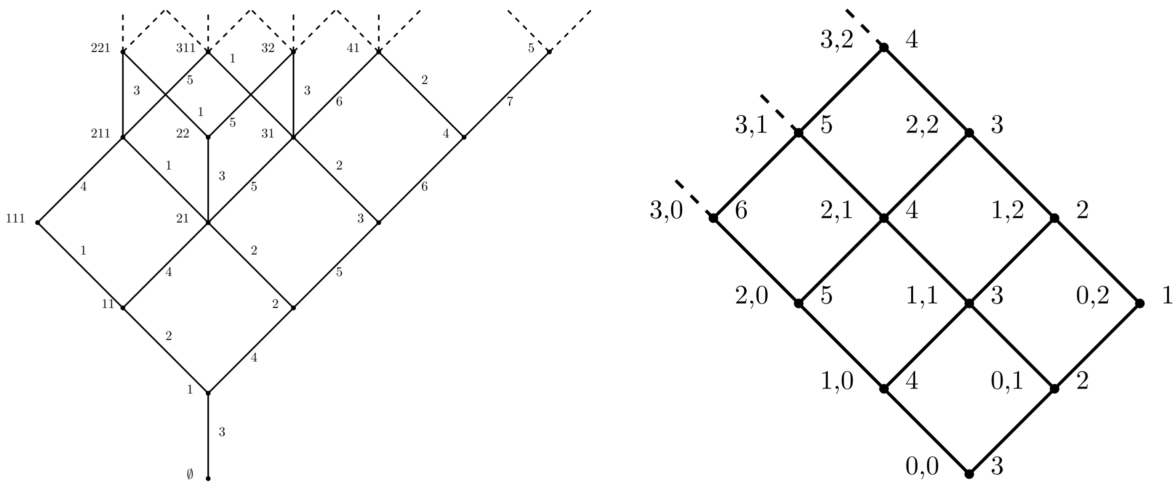
Figure 2. The shifted Young's lattice $\mathbb{SY}$, the lattice of strict partitions, with its canonical weighting.

3. Cartesian product

It is straightforward to show that cartesian product preserves Fomin lattices and that it preserves our other criteria. Surprisingly, the converse is also true.

Theorem 3.1. *If L is a Fomin lattice with differential degree r and weighting w, and L is the cartesian product of two lattices $L = L_1 \times L_2$, then*

(1) L_1 is a Fomin lattice with some differential degree r_1 and weighting w_1.
(2) L_2 is a Fomin lattice with some differential degree r_2 and weighting w_2.
(3) $r = r_1 + r_2$.
(4) For any $x \lessdot y$ in L_1 and z in L_2, $w((x, z) \lessdot (y, z)) = w_1(x \lessdot y)$.



(A) Lattice $\mathbb{Y}_3$ with weights of the coverings (B) $J(\mathbb{Y}_3)$ with weights of the points

FIGURE 3. The lattice $\mathbb{Y}_3$ of partitions with ≤ 3 rows, with its canonical weighting.

(5) For any w in L_1 and $x \lessdot y$ in L_2, $w((v,x) \lessdot (v,y)) = w_2(x \lessdot y)$.

Proof. First we show that for any $x \lessdot y$ in L_1 and any $z_1, z_2 \in L_2$,

$$w((x,z_1) \lessdot (y,z_1)) = w((x,z_2) \lessdot (y,z_2)). \tag{2}$$

Because of property (E), w is constant on cover-projective pairs of elements of L, so equation (2) holds if $z_1 \lessdot z_2$. Since any two elements $z_1, z_2 \in L_2$ can be connected by a finite chain of upward and downward covers, equation (2) holds for any z_1 and z_2. Thus the w_1 in conclusion (4) exists.

Similarly, for any $w_1, w_2 \in L_1$ and any $x \lessdot y$ in L_2

$$w((w_1,x) \lessdot (w_1,y)) = w((w_2,x) \lessdot (w_2,y)). \tag{3}$$

Thus the w_2 in conclusion (5) exists.

We now show that r_1 and r_2 exist, that is, that L_1 is a Fomin lattice with differential degree r_1 and weighting w_1 and similarly for L_2. Consider any $x \in L_1$ and any $z_1, z_2 \in L_2$. Equation (1) for (x, z_1) is:

$$\sum_{(p,q)\,|\,(p,q)\lessdot(x,z_1)} w((p,q) \lessdot (x,z_1)) + r = \sum_{(t,u)\,|\,(x,z_1)\lessdot(t,u)} w((x,z_1) \lessdot (t,u))$$

$$\sum_{p\,|\,p\lessdot x} w((p,z_1) \lessdot (x,z_1)) + \sum_{q\,|\,q\lessdot z_1} w((x,q) \lessdot (x,z_1)) + r = \sum_{t\,|\,x\lessdot t} w((x,z_1) \lessdot (t,z_1)) + \sum_{u\,|\,z_1\lessdot u} w((x,z_1) \lessdot (x,u))$$

$$\sum_{p\,|\,p\lessdot x} w_1(p \lessdot x) + \sum_{q\,|\,q\lessdot z_1} w_2(q \lessdot z_1) + r = \sum_{t\,|\,x\lessdot t} w_1(x \lessdot t) + \sum_{u\,|\,z_1\lessdot u} w_2(z_1 \lessdot u) \tag{4}$$

Similarly, equation (1) for (x, z_2) is:

$$\sum_{p\,|\,p\lessdot x} w_1(p \lessdot x) + \sum_{q\,|\,q\lessdot z_2} w_2(q \lessdot z_2) + r = \sum_{t\,|\,x\lessdot t} w_1(x \lessdot t) + \sum_{u\,|\,z_2\lessdot u} w_2(z_2 \lessdot u) \tag{5}$$

Subtracting (5) from (4) gives

$$\sum_{q\,|\,q\lessdot z_1} w_2(q \lessdot z_1) - \sum_{q\,|\,q\lessdot z_2} w_2(q \lessdot z_2) = \sum_{u\,|\,z_1\lessdot u} w_2(z_1 \lessdot u) - \sum_{u\,|\,z_2\lessdot u} w_2(z_2 \lessdot u)$$

$$\sum_{u\,|\,z_1\lessdot u} w_2(z_1 \lessdot u) - \sum_{q\,|\,q\lessdot z_1} w_2(q \lessdot z_1) = \sum_{u\,|\,z_2\lessdot u} w_2(z_2 \lessdot u) - \sum_{q\,|\,q\lessdot z_2} w_2(q \lessdot z_2)$$

This shows that $\sum_{u\,|\,z\lessdot u} w_2(z \lessdot u) - \sum_{q\,|\,q\lessdot z} w_2(q \lessdot z)$ is independent of $z \in L_1$. We define r_2 as this value.

Similarly, we show that $\sum_{t \mid z \lessdot t} w_1(z \lessdot t) - \sum_{q \mid q \lessdot z} w_1(q \lessdot z)$ is independent of $z \in L_1$, and we define r_1 as this value.

From these facts it is straightforward to prove the conclusions of the theorem. □

Lemma 3.2. *If L is a Fomin lattice with $\hat{0}_L$, L is not the trivial (one-element) lattice, $V = \mathbb{Z}$, and all values of w are positive, then r is positive.*

Proof.

For $x = \hat{0}_L$, (1) becomes $r = \sum_{z \mid x \lessdot z} w(\hat{0}_L \lessdot z)$. But by hypothesis, there is at least one z in the range of the sum and $w(\hat{0}_L \lessdot z) > 0$, which ensures the sum is positive. That proves r is positive. □

Lemma 3.3. *If L is a Fomin lattice with differential degree r and weighting w, then*

(1) If L has a minimum element $\hat{0}_L$, then L_1 has a minimum element $\hat{0}_{L_1}$ and L_2 has a minimum element $\hat{0}_{L_2}$.
(2) If L has a minimum element $\hat{0}_L$ and is positive, and L_1 and L_2 are not the trivial lattice, then L_1 and L_2 are positive.
(3) If L is distributive, then L_1 and L_2 are distributive.

Proof.

Regarding (1) and (3): These are straightforward lattice reasoning.

Regarding (2): By the construction in th. 3.1, L_1 has some differential degree r_1 and weighting w_1 and L_2 has differential degree r_2 and weighting w_2. Every value of w_1 and every value of w_2 are values of w for suitable arguments, so all values of w_1 and w_2 are positive integers. Since L has a $\hat{0}_L$, L_1 has a $\hat{0}_{L_1}$ and L_2 has a $\hat{0}_{L_2}$. By lem. 3.2, both r_1 and r_2 are positive. Thus L_1 and L_2 are positive. □

Remark 3.4.

(1) The trivial lattice is a Fomin lattice: Its w must be the empty function and $r = 0$. It is the identity of the cartesian product of Fomin lattices. By our definition, this is not a positive weighting. Thus lem. 3.3(2) requires the hypothesis that L_1 and L_2 are not the trivial lattice.[4]

(2) Lem. 3.3(2) requires the hypothesis that $\hat{0}_L$ exists: Consider $\mathbb{Y}$, the lattice of partitions, which is a positive Fomin lattice with $w = 1$ and $r = 1$. Consider $\mathbb{Y}^*$, the lattice dual of $\mathbb{Y}$, which is a Fomin lattice with $w = 1$ and $r = -1$ and so is not positive. Define $2\mathbb{Y}$ to be the Fomin lattice $\mathbb{Y}$ with $w = 2$ and $r = 2$, which is positive. Then $L = 2\mathbb{Y} \times \mathbb{Y}^*$ is a Fomin lattice with $w = 1$ or 2 and $r = 1$, which is positive. Thus $L = 2\mathbb{Y} \times \mathbb{Y}^*$ is a positive Fomin lattice that is a product of Fomin lattices, one of which is not positive.

Lemma 3.5. *Let P be a poset and $P = P_1 \sqcup P_2$, the disjoint union of two posets P_1 and P_2. Then if P is unique-cover-modular, then P_1 and P_2 are unique-cover-modular. (def. 2.1.)*

Remark 3.6. *Suppose L is a distributive lattice, L is finitary, and L has finite upward covers. It is not necessary that P has finite upward covers, as is shown by the counterexample shown in fig. 4.*

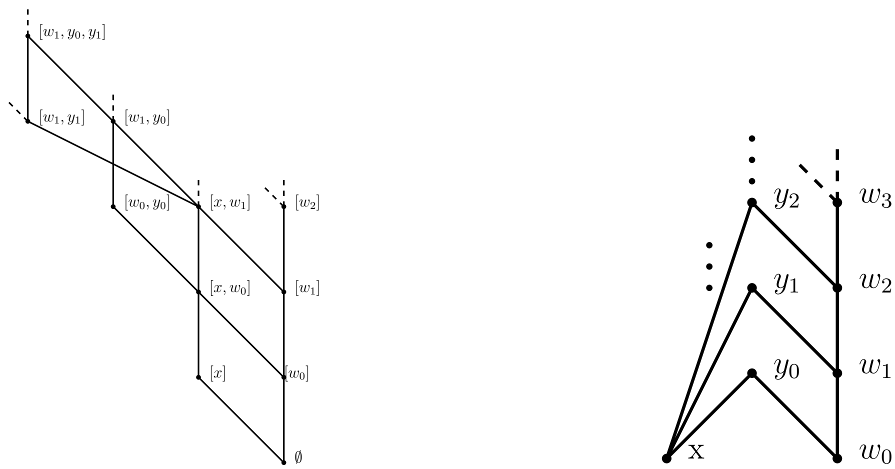
Define P to be the poset with elements $x, y_0, y_1, y_2, \ldots, w_0, w_1, w_2, \ldots$ and the coverings:

- $w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots$,
- $x \lessdot y_i$ for all $i \geq 0$, *and*
- $w_i \lessdot y_i$ for all $i \geq 0$.

Define L to be the lattice $L = \mathrm{Ideal_f}(P)$ of all finite ideals of P. By Birkhoff's Representation Theorem[Birk1967, §III.3 Th. 3][Stan2012, Prop. 3.4.3] *L is distributive, L is finitary, and P is poset-isomorphic to the set of join-irreducible elements of L. The elements of L are (where $[a, b, c, \ldots]$ is the ideal of P generated by the elements $a, b, c, \ldots$):*

- $[w_i] = \{w_0, w_1, w_2, \ldots, w_i\}$ for all $i \geq -1$ *(generating $\emptyset$ when $i = -1$),*
- $[x, w_i] = \{x, w_0, w_1, w_2, \ldots, w_i\}$ for all $i \geq -1$ *(generating $\{x\}$ when $i = -1$), and*
- $[w_i, y_{n_1}, y_{n_2}, y_{n_3}, \ldots y_{n_j}] = \{x, w_0, w_1, w_2, \ldots, w_i, y_{n_1}, y_{n_2}, y_{n_3}, \ldots y_{n_j}\}$ for all $i \geq 0$, all $j \geq 1$, *and* all $0 \leq n_1 < n_2 < n_3 < \cdots < n_j \leq i$ *(some non-empty finite set of $y_\bullet$ added to a compatible ideal of the preceding type),*

[4]Though our presentation of the theory might be made more uniform if we defined the trivial Fomin lattice to be positive.



(A) Lattice L has finite upward covers (B) Poset $P = J(L)$ does not have finite upward covers

Figure 4. Counterexample in rem. 3.6

We see that each of the above types of elements of L (ideals of P) has only a finite number of ideals that cover it in L, that is, contain one more element than it does. So L has finite upward covers. We see that x has all of the $y_\bullet$ as upward covers in P. So P does not have finite upward covers.

4. Local characterization

We start with proving various "local" facts about the poset P of points (join-irreducible elements) of L. These culminate with a classification of the possible structures of the neighborhood of a point p, that is, the convex set generated by the points that cover of p and the points that are covered by p.

4.1. Consequences of distributivity.

Theorem 4.1. [Birk1967, §III.3 Th. 3][Stan2012, Prop. 3.4.3] *Since we have assumed that $\hat{0}_L$ exists, L is finitary, that is, all if its principal ideals are finite. L is isomorphic to the set of finite order ideals of the poset $P = J(L)$ of its join-irreducible elements, which we denote $L = \mathrm{Ideal_f}(P)$. Given a covering $x \lessdot y$ in L, $x \setminus y$ (considered as the set difference of two ideals of P) contains exactly one element of P, and the projectivity requirement (condition (A)(5)) on w can be simplified to: There is a weighting function w from P into V, and for every $x \lessdot y$ in L, $w(x \lessdot y) = w(y \setminus x)$.*

The weighted-differential condition (1) is equivalent to

$$\sum_{a \,|\, a \text{ maximal in } x} w(a) + r = \sum_{b \,|\, b \text{ minimal in } P \setminus x} w(b) \qquad \text{for all } x \in \mathrm{Ideal_f}(P). \tag{6}$$

Because the image of w on L is the same as the image of w on P, we abuse language by conflating the two meanings of w when we are dealing them as entire functions. In particular, each of them has all of its values positive integers iff the other does, and so "w is positive" is used to mean both.

Definition 4.2. *The elements of L are called* diagrams*. The* size *of a diagram is its rank in L. The elements of P are called* points *or* boxes*.*

Lemma 4.3.

(1) *P is locally finite.*
(2) *P is finitary.*

Proof.

Regarding (1): If there was some $a \leq b$ in P for which $[a, b]$ is infinite, then the set of all (closed) principal ideals of the elements in $[a, b]$, $\{[x] \,|\, a \leq x \leq b\}$, would be an infinite subset of L with all of its elements $\geq [a]$ and $\leq [b]$. That contradicts that L is locally finite.

Regarding (2): Similarly, if there was an $a \in P$ whose principal ideal $[a]$ in P was infinite, then $\{[x] \,|\, x \leq b\}$ would be an infinite subset of L with all of its elements $\leq [a]$. That contradicts that, given our criteria, th. 4.1 shows that L is finitary. ☐

Lemma 4.4. *The lattice of diagrams is isomorphic to the lattice of finite ideals of points under union and intersection, the size of a diagram is the cardinality of its corresponding ideal (the number of points it contains), and the number of tableaux with a particular shape is the number of linear orderings of its points (a subposet of P).*

Definition 4.5. *Given any diagram x, its* insertion points *are the points that are the minimal elements of $P \setminus x$, those which can be added to x to make a diagram (whose size is 1 larger than the size of x). The* deletion points *of x are the points that are the maximal elements of x, those which can be deleted from x to make a diagram (whose size is 1 smaller than the size of x). For any diagram x, we define D_x to be the set of deletion points of x, and I_x to be the set of insertion points of x.*

The summation on the left of (6) is over the deletion points of x and the summation on the right of (6) is over the insertion points of x. Thus, (6) is equivalent to

$$\sum_{a \in D_x} w(a) + r = \sum_{b \in I_x} w(b) \qquad \text{for all } x \in \mathrm{Ideal_f}(P). \tag{7}$$

Applying (6) to the ideal $\emptyset$ gives

$$r = \sum_{z \,|\, z \text{ minimal in } P} w(z) \tag{8}$$

In many cases P has a unique minimal element, $\hat{0}_P$, and in that case, (8) reduces to $w(\hat{0}_P) = r$.

Given an ideal x and a point p in I_x, that is, minimal in $P \setminus x$, we can construct the ideal $x' = x \sqcup \{p\}$. Thus $P \setminus x = (P \setminus x') \sqcup \{p\}$ and $p \in D_{x'}$.

The difference between (7) applied to x' and (7) applied to x is

$$\sum_{a \in D_{x'} \setminus D_x} w(a) - \sum_{b \in D_x \setminus D_{x'}} w(b) = \sum_{a \in I_{x'} \setminus I_x} w(a) - \sum_{b \in I_x \setminus I_{x'}} w(b) \qquad \text{for all } x \in \mathrm{Ideal_f}(P),\ p \in I_x. \tag{9}$$

The domains of the four summations of (9) are:

(1) $D_{x'} \setminus D_x$: The set of maximal elements of x' that are not maximal elements of x, which is $\{p\}$,
(2) $D_x \setminus D_{x'}$: The set of maximal elements of x that are not maximal elements of x',
(3) $I_{x'} \setminus I_x$: The set of minimal elements of $P \setminus x'$ that are not minimal elements of $P \setminus x$,
(4) $I_x \setminus I_{x'}$: The set of minimal elements of $P \setminus x$ that are not minimal elements of $P \setminus x'$, which is $\{p\}$.

Lemma 4.6. *Condition (6) (for every $x \in \mathrm{Ideal_f}(P)$) is equivalent to the combination of (8) and (9) (for every $x \in \mathrm{Ideal_f}(P)$ and $p \in I_x$).*

Proof.

Regarding $\Rightarrow$: This has been proved by the above derivation of (8) and (9) from (6).

Regarding $\Leftarrow$: For any $y \in \mathrm{Ideal_f}(P)$, we want to prove (6) for its $x = y$. Since y is finite, we can prove this by induction upward on the size of y. The base case is $y = \emptyset = \hat{0}_L$. In that case, (6) with its $x = \hat{0}_L$ reduces to (8).

If y has positive size, let $y' \lessdot y$ in $\mathrm{Ideal_f}(P)$. By induction, (6) is true for its $x = y'$. Now consider (9) with its $x = y'$ and its p the unique element of $y \setminus y'$. We can add these two equations and by the above derivation obtain (6) for its $x = y$. ☐

Definition 4.7. *Given a point $p \in P$, define C_p^- (C^- for "downward covers") as the set of all elements of P which are covered by p. Similarly, define C_p^+ (C^+ for "upward covers") as the set of all elements of P which cover p. Define N_p (N for "neighborhood") to be $\{x \in P \,|\, (\exists a \in C_p^-, b \in C_p^+)\, a \leq x \leq b\}$, the convex subset of P generated by C_p^- and C_p^+. Define S_p (S for "siblings") to be $\{x \in P \,|\, x \neq p \text{ and } (\exists a \in C_p^-, b \in C_p^+)\, a < x < b\} = N_p \setminus C_p^- \setminus C_p^+ \setminus \{p\}$, N_p with C_p^-, C_p^+, and p deleted.*

Lemma 4.8. *By construction, every element of C_p^+ is $>$ every element of C_p^-, but no element of C_p^+ covers any element of C_p^-. All elements of C_p^+ are incomparable to each other. Dually for C_p^-. Every element of S_p is incomparable to p. If P is graded, every element of S_p has the same grade as p and S_p is an antichain. However, in general, S_p need not be an antichain.*

Lemma 4.9. *Given an ideal x and a point p that is an insertion point of x, $C_p^- \subset x$ and $C_p^+ \subset P \setminus (x \sqcup \{p\})$.*

Definition 4.10. *Given an ideal x and a point p in I_x, define $S_{px}^- = S_p \cap x$ and $S_{px}^+ = S_p \cap (P \setminus (x \sqcup \{p\}))$.*

Lemma 4.11. *Since $p \notin S$, by construction, $S_p = S_{px}^- \sqcup S_{px}^+$, and if $s^- \in S_{px}^-$ and $s^+ \in S_{px}^+$ are comparable, then $s^- \leq s^+$. So S_{px}^- is an ideal of S_p and S_{px}^+ is a filter of S_p.*

The generic relationships of the elements of these sets are shown in fig. 5.

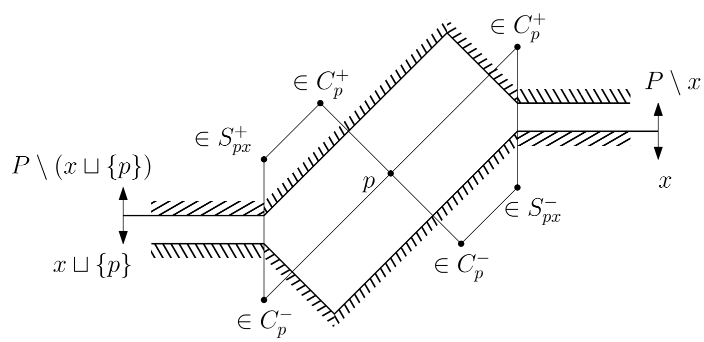


FIGURE 5. Adding a point p to an ideal x making an ideal x'

4.2. Consequences of unique-cover-modularity.

Lemma 4.12. *Let $p \in P$. For any $a \in C_p^+$ and any $b \neq p$ that is covered by a, unique-cover-modularity shows that there is a $c \in C_p^-$ that is covered by a and so $b \in S_p$. Dually, for any $a \in C_p^-$ and any $b \neq p$ that covers a, there is a $c \in C_p^+$ that covers a and so $b \in S_p$. For any $a \in S_p$, unique-cover-modularity shows there is a unique element of C_p^+ which is $> a$. Dually, there is a unique element of C_p^- which is $< a$.*

Definition 4.13. *For any $p \in P$ and any $a \in S_p$, we define a_p^+ to be the unique element of C_p^+ which is $> a$. Dually, we define a_p^- to be the unique element of C_p^- which is $< a$.*

Lemma 4.14. *The four summation domain sets of (9) are:*

(1) $D_{x'} \setminus D_x = \{p\}$
(2) $D_x \setminus D_{x'}$ is the set of maximal elements of x which are covered by p. This is the same as the elements of C_p^- which are not $\leq$ (or equivalently, not covered by) any element of S_{px}^-. We denote this property by $\not\leq S_{px}^-$.
(3) $I_{x'} \setminus I_x$ is the set of minimal elements of $P \setminus x'$ which cover p. This is the same as the elements of C_p^+ which are not $\geq$ (or equivalently, do not cover) any element of S_{px}^+. We denote this property by $\not\geq S_{px}^+$.
(4) $I_x \setminus I_{x'} = \{p\}$

This shows (9) is equivalent to

$$w(p) - \sum_{a \mid a \in C_p^-,\ a \not\leq S_{px}^-} w(a) = \sum_{b \mid b \in C_p^+,\ b \not\geq S_{px}^+} w(b) - w(p) \qquad \text{for all } x \in \mathrm{Ideal_f}(P),\ p \in I_x$$

or equivalently

$$\sum_{a \mid a \in C_p^-,\ a \not\leq S_{px}^-} w(a) + \sum_{b \mid b \in C_p^+,\ b \not\geq S_{px}^+} w(b) = 2w(p) \qquad \text{for all } x \in \mathrm{Ideal_f}(P),\ p \in I_x. \tag{10}$$

Proof.

Regarding (1) and (4): These are straightforward.

Regarding (2) $\Rightarrow$: Let $a \in D_x \setminus D_{x'}$. Then a is maximal in x but not maximal in x'. Thus $a \leq$ an element of x' which is not an element of x. The only element of x' that is not an element of s is p, so $a < p$. If a is not covered by p, then there is a b with $a < b < p$. Thus $b \in x'$ but since a is maximal in x, $b \notin x$, which requires that $b = p$, which is a contradiction. Thus a is covered by p and so $a \in C_p^-$.

To show that $a \not\lessdot S^-_{px}$, assume there is a c in S^-_{px} with $a \leq c$. Since C^-_p is disjoint from S^-_{ps}, $a < c$. But $c \in x$, contradicting that a is maximal in x. Thus $a \not\lessdot S^-_{px}$.

Regarding (2) $\Leftarrow$: Let $a \in C^-_p$ and $a \not\lessdot S^-_{px}$. Because $a < p$ and $p \in x'$, a is not maximal in x'.

To show that a is maximal in x, assume there is a d in x with $d > a$. We can assume that $d \gtrdot a$. Then we apply unique-cover-modularity to p and d to obtain a e which covers p and d. So $e \in C^+_p$, and with $a \in C^-_p$ shows $d \in S_p$. Since $d \in x$, $d \in S^-_{px}$. This contradicts that $a \not\lessdot S^-_{ps}$, so a is maximal in x.

Regarding (3): This is proved dually to (2). ☐

4.3. Reorganizing the linear equations. Looking at (10) as a set of linear equations in the values of w, we see that each equation is determined by choosing first an $x \in \mathrm{Ideal_f}(P)$ and then a $p \in I_x$. We will now reorganize the statement of the set of linear equations (10), but indexed first by an arbitrary choice of $p \in P$ (which determines C^+_p and C^-_p), and then a choice of values of S^+_{px} and S^-_{px} consistent with p.

Lemma 4.15. *The set of equations (10) is the same set as the equations*

$$\sum_{a \in C^-_p,\ a \not\lessdot T^-} w(a) + \sum_{b \in C^+_p,\ b \not\gtrdot T^+} w(b) = 2w(p) \qquad \text{for all } p \in P,\ S_p = T^- \sqcup T^+, \text{ with } T^- \text{ an ideal and } T^+ \text{ a filter.} \tag{11}$$

Proof.

A particular equation in the set (10) is indexed first by an $x \in \mathrm{Ideal_f}(P)$ and then a $p \in I_x$. p determines C^+_p and C^-_p, and p and x together determine S^-_{px} and S^+_{px}. Necessarily, $S_p = S^+_{px} \sqcup S^+_{px}$, so setting $T^- = S^-_{px}$ and $T^+ = S^+_{px}$ shows that the equation is in the set (11).

Conversely, a particular equation in the set (11) is indexed first by a $p \in P$ and then a partition of S_p into an ideal T^- and a filter T^+. Define $x = \{z \in P \,|\, z < p\} \sqcup T^- = (p) \sqcup T^-$, the principal ideal of P generated by p, less p itself, plus T^-. By construction, x is an ideal of P, p is in insertion point of x, $S^-_{px} = T^-$ and $S^+_{px} = T^+$. This shows that the equation is in the set (10). ☐

4.4. Orphans.

Definition 4.16. *Define $x \in C^+_p$ as an* upward orphan *(relative to p) if x does not cover any element of S_p, which is equivalent to x covering only one element of N_p (namely p). Similarly, we define $x \in C^-_p$ as a* downward orphan *(relative to p) if x is not covered by any element of S_p, which is equivalent to x being covered by only one element of N_p (namely p).*

In (11), if we choose $T^- = \emptyset$ and $T^+ = S_p$, it becomes

$$\sum_{a \in C^-_p} w(a) + \sum_{b \in C^+_p,\ b \not\gtrdot S_p} w(b) = 2w(p) \qquad \text{for all } p \in P$$

By lem. 4.12, S_p contains every element covered by any element of C^+_p except for p, so the second sum is over all upward orphans. Thus

Lemma 4.17.

$$\sum_{a \,|\, a \lessdot p} w(a) + \sum_{b \,|\, b \gtrdot p,\ b \text{ upward orphan}} w(b) = 2w(p) \qquad \text{for all } p \in P \tag{12}$$

Equation (12) allows the weight of a point to be computed as half of the sum of the weights of the elements it covers, if the point is not covered by any upward orphans.

If we choose $T^- = S_p$ and $T^+ = \emptyset$, then (11) becomes

Lemma 4.18.

$$\sum_{a \,|\, a \lessdot p,\ a \text{ downward orphan}} w(a) + \sum_{b \,|\, b \gtrdot p} w(b) = 2w(p) \qquad \text{for all } p \in P \tag{13}$$

4.5. Partitioning S into three sets. Given $p \in P$, let us partition S_p into three disjoint parts $S_p = L \sqcup \{m\} \sqcup U$ where L is an ideal of S, U is a filter of S, and thus m is an element between L and U. In (11), setting its $T^- = L$ and its $T^+ = \{m\} \sqcup U$ gives

$$\sum_{a \,|\, a \in C_p^-,\ a \not\leq L} w(a) + \sum_{b \,|\, b \in C_p^+,\ b \not\geq \{m\} \sqcup U} w(b) = 2w(p) \tag{14}$$

and then setting its $T^- = L \sqcup \{m\}$ and its $T^+ = U$ in (11) gives

$$\sum_{a \,|\, a \in C_p^-,\ a \not\leq L \sqcup \{m\}} w(a) + \sum_{b \,|\, b \in C_p^+,\ b \not\geq U} w(b) = 2w(p) \tag{15}$$

Since $x \not\leq L \sqcup \{m\}$ implies $x \not\leq L$ and $x \not\geq \{m\} \sqcup U$ implies $x \not\geq U$, subtracting (15) from (14) gives

$$\sum_{a \,|\, a \in C_p^-,\ a \not\leq L,\ \text{not } a \not\leq L \sqcup \{m\}} w(a) = \sum_{b \,|\, b \in C_p^+,\ b \not\geq U,\ \text{not } b \not\geq \{m\} \sqcup U} w(b)$$

This is equivalent to

$$\sum_{a \,|\, a \in C_p^-,\ a \not\leq L,\ a \leq m} w(a) = \sum_{b \,|\, b \in C_p^+,\ b \not\geq U,\ b \geq m} w(b) \tag{16}$$

Lemma 4.19. *Given a $p \in P$, equation (11) holds for every partition of S_p into an ideal T^- and a filter T^+ iff:*

(1) equation (12) holds for p and
(2) equation (16) holds for p and for every partition of S_p into three disjoint sets $S_p = L \sqcup \{m\} \sqcup U$, where L is an ideal of S_p and U is a filter of S_p.

Proof.
Regarding $\Rightarrow$: This has been proved by the above derivations of (12), (13), and (16).

Regarding $\Leftarrow$: For a given p, we prove (11) for any partition $S_p = T^- \sqcup T^+$ where T^- is an ideal and T^+ is a filter by induction upward on the size of T^-. If $T^- = \emptyset$, then by hypothesis (12) holds for p, which is equivalent to (11) for $T^- = \emptyset$.

If $T^- \neq \emptyset$, then choose m as a maximal element of T^- and define $L = T^- \setminus \{m\}$ and $U = T^+$, so that $S_p = L \sqcup \{m\} \sqcup U$ with L being an ideal and U being a filter. Since $|L| = |T^-| - 1$, by induction (11) holds for the partition $S_p = L \sqcup (\{m\} \sqcup U)$, which is

$$\sum_{a \in C_p^-,\ a \not\leq L} w(a) + \sum_{b \in C_p^+,\ b \not\geq \{m\} \sqcup U} w(b) = 2w(p) \tag{17}$$

By hypothesis, (16) holds for the partition $S = L \sqcup \{m\} \sqcup U$, which is

$$\sum_{a \,|\, a \in C_p^-,\ a \not\leq L,\ a \leq m} w(a) = \sum_{b \,|\, b \in C_p^+,\ b \not\geq U,\ b \geq m} w(b) \tag{18}$$

Subtracting the left side of (18) from the first term of (17) and adding the right side of (18) to the second term of (17) gives

$$\sum_{a \in C_p^-,\ a \not\leq L,\ a \not\leq m} w(a) + \sum_{b \in C_p^+,\ b \not\geq U} w(b) = 2w(p)$$

This equation is the same as (11) for the partition $S_p = (L \sqcup \{m\}) \sqcup U = T^- \sqcup T^+$.

Thus by induction (11) holds for p and every partition of S_p into an ideal T^- and a filter T^+. □

We can replace (12) with (13):

Lemma 4.20. *Given a $p \in P$, equation (11) holds for every partition of S_p into an ideal T^- and a filter T^+ iff:*

(1) equation (13) holds for p and
(2) equation (16) holds for p and for every partition of S_p into three disjoint sets $S_p = L \sqcup \{m\} \sqcup U$, where L is an ideal of S_p and U is a filter of S_p.

Proof.
This is proved dually to the proof of lem. 4.19, using (13) in place of (12). □

4.6. Consequences of positive weighting.

Lemma 4.21. *For any $p \in P$ and for any $x, y \in S_p$, if $x_p^+ = y_p^+$ and $x_p^- = y_p^-$, then $x = y$.*

Proof.
Assume there is $x, y \in S_p$, with $x \neq y$, $x_p^+ = y_p^+$, and $x_p^- = y_p^-$. Since $x \neq y$, either $x \not\leq y$ or $y \not\leq x$. Without loss of generality, assume $y \not\leq x$. Define $X = \{s \in S_p \,|\, s_p^- = x_p^- \text{ and } s_p^+ = x_p^+ \text{ and } s \leq x\} = [x] \cap [x_p^-, x_p^+]$, the principal ideal generated by x within the interval $[x_p^-, x_p^+]$. By construction, $x \in X$ and $y \notin X$.

Given $a \in C_p^-$, $a \leq X$ iff $a \leq x$, so $a \not\leq X$ iff $a \not\leq x$. Given $b \in C_p^+$, if $b \geq$ some $z \in X$, $z_p^+ = b$. But since $z \leq x$, lem. 4.12 shows $z_p^+ = x_p^+$, so $b = z_p^+ = x_p^+ = y_p^+$ and $b \geq y \notin X$. Thus $b \geq S_p$ iff $b \geq S_p \setminus X$ and $b \not\geq S_p$ iff $b \not\geq S_p \setminus X$.

Applying (11) to $S_p = \emptyset \sqcup S_p$ gives

$$\sum_{a \in C_p^-,\ a \not\leq \emptyset} w(a) + \sum_{b \in C_p^+,\ b \not\geq S_p} w(b) = 2w(p)$$

which is equivalent to

$$\sum_{a \in C_p^-} w(a) + \sum_{b \in C_p^+,\ b \not\geq S_p \setminus X} w(b) = 2w(p) \tag{19}$$

Applying (11) to $S_p = X \sqcup (S_p \setminus X)$ gives

$$\sum_{a \in C_p^-,\ a \not\leq X} w(a) + \sum_{b \in C_p^+,\ b \not\geq S_p \setminus X} w(b) = 2w(p)$$

which is equivalent to

$$\sum_{a \in C_p^-,\ a \not\leq x} w(a) + \sum_{b \in C_p^+,\ b \not\geq S_p \setminus X} w(b) = 2w(p) \tag{20}$$

Subtracting (20) from (19) gives $w(x_p^-) = 0$. By the criterion that w is positive, this is a contradiction. Thus $x = y$. ☐

Lemma 4.22. *For any $p \in P$, S_p is an antichain.*

Proof.
Assume there exists $x < y$ in S_p. By lem. 4.12, $x_p^+ = y_p^+$ and $x_p^- = y_p^-$. By lem. 4.21, $x = y$, which is a contradiction. Thus S_p is an antichain. ☐

Thus, for any p, any subset of S_p is both an ideal and a filter, which allows (11) and (16) to be applied quite broadly. Consider how unique-cover-modularity simplifies (16): For any given $m \in S_p$, any a on the left-hand side of (16) can only be m_p^-. Similarly, any b on the right-hand side can only be m_p^+. Thus (16) is equivalent to:

$$\left.\begin{matrix} \text{if } m_p^- \not\leq L & w(m_p^-) \\ \text{otherwise} & 0 \end{matrix}\right\} = \begin{cases} w(m_p^+) & \text{if } m_p^+ \not\geq U \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

If we set $L = \emptyset$ and $U = S_p \setminus \{m\}$,

$$w(m_p^-) = \begin{cases} w(m_p^+) & \text{if } m_p^+ \not\geq U \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Given the criterion of unique-cover-modularity, $m_p^+ \geq z$ for some $z \in U$ iff $m_p^+ = z_p^+$ for some $z \in U$. So $m_p^+ \not\geq U$ iff there is no $z \in U$ with $m_p^+ = z_p^+$. Given the criterion of positivity, $w(m_p^-) > 0$, which is contradicted by (22) unless there is no $z \in S_p$ with $z_p^+ = m_p^+$ and $z \neq m$.

Dually, if we set $L = S_p \setminus \{m\}$ and $U = \emptyset$, we get a contradiction unless there is no $z \in S_p$ with $z_p^- = m_p^-$ and $z \neq m$.

Thus for every $m \neq n$ in S_p, $m_p^- \neq n_p^-$ and $m_p^+ \neq n_p^+$, and so the set of pairs $\{(m_p^-, m_p^+) \,|\, m \in S_p\}$ is a bijection between subsets of C_p^- and C_p^+. In addition, (21) reduces to

$$w(m_p^-) = w(m_p^+) \qquad \text{for all } p \in P \text{ and } m \in S_p \tag{23}$$

Lemma 4.23. *Given $p \in P$, (16) holds for every partition of S_p into three disjoint sets $S_p = L \sqcup \{m\} \sqcup U$ with L an ideal of S_P and U a filter of S_p iff the following conditions are true:*

1. *the set of pairs $\{(m_p^-, m_p^+) \,|\, m \in S_p\}$ is a bijection between subsets of C_p^- and C_p^+ and*
2. *for every $m \in S_p$, $w(m_p^-) = w(m_p^+)$.*

Proof.
Regarding $\Rightarrow$: This has been shown by the above derivations.
Regarding $\Leftarrow$: Assume $S_p = L \sqcup \{m\} \sqcup U$. By the above derivation, (16) is the equivalent to (21). By hypothesis (1) and lem. 4.12, $m_p^- \leq z$ for some $z \in L$ iff $m_p^- = z_p^-$ iff $m = z$. But by construction, $m \notin L$, so $m_p^- \not\leq L$ is always true. Dually, $m_p^+ \not\leq U$ is always true. Thus (21) is equivalent to $w(m_p^-) = w(m_p^+)$ which is assumed as hypothesis (2). ☐

At this point, let us summarize our progress:

Lemma 4.24. *Given*

L satisfies the criteria of a Fomin lattice other than (A)(4), the weighted-differential property,
L has a $\hat{0}_L$,
L is distributive,
P is unique-cover-modular, and
$w(\bullet)$ from P into $V = \mathbb{Z}$ has only positive values,

then L satisfies (9) (and thus is a Fomin lattice) iff

(eq. 8) $r = \sum_{z \,|\, z \text{ minimal in } P} w(z)$,
(eq. 12) $\sum_{a \,|\, a \lessdot p} w(a) + \sum_{b \,|\, b \gtrdot p,\ b \text{ upward orphan}} w(b) = 2w(p)$ for all $p \in P$,
(lem. 4.23(1)) the set of pairs $\{(m_p^-, m_p^+) \,|\, m \in S_p\}$ is a bijection between subsets of C_p^- and C_p^+ for all $p \in P$, and
(lem. 4.23(2)) $w(m_p^-) = w(m_p^+)$ for all $p \in P$, $m \in S_p$.

Proof.
This is proved by chaining together th. 4.1, lem. 4.6, lem. 4.14, lem. 4.15, lem. 4.19, and lem. 4.23. ☐

Similarly to lem. 4.19 and 4.20, we can replace eq. (12) with eq. (13):

Lemma 4.25. *Given*

L satisfies the criteria of a Fomin lattice other than (A)(4), the weighted-differential property,
L has a $\hat{0}_L$,
L is distributive,
P is unique-cover-modular, and
$w(\bullet)$ from P into $V = \mathbb{Z}$ has only positive values,

then L satisfies (9) (and thus is a Fomin lattice) iff

(eq. 8) $r = \sum_{z \,|\, z \text{ minimal in } P} w(z)$,
(eq. 13) $\sum_{a \,|\, a \lessdot p,\ a \text{ downward orphan}} w(a) + \sum_{b \,|\, b \gtrdot p} w(b) = 2w(p)$ for all $p \in P$,
(lem. 4.23(1)) the set of pairs $\{(m_p^-, m_p^+) \,|\, m \in S_p\}$ is a bijection between subsets of C_p^- and C_p^+ for all $p \in P$, and
(lem. 4.23(2)) $w(m_p^-) = w(m_p^+)$ for all $p \in P$, $m \in S_p$.

Proof.
This is proved by chaining together th. 4.1, lem. 4.6, lem. 4.14, lem. 4.15, lem. 4.20, and lem. 4.23. ☐

4.7. No upward or downward triple covers.

Lemma 4.26. *No element of P is covered by three or more distinct elements.*

Proof.
The construction for this poof is shown in fig. 6.

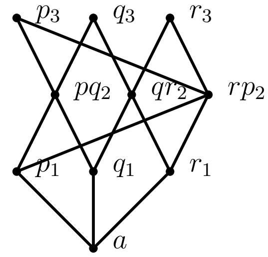


FIGURE 6. Construction for the proof of lem. 4.26

Assume that we have $a \in P$ covered by distinct p_1, q_1 and r_1. By unique-cover-modularity, there is a pq_2 covering p_1 and q_1, a qr_2 covering q_1 and r_1, and an rp_2 covering r_1 and p_1. By lem. 4.21 (applied to p_1, q_1, and r_1), pq_2, qr_2, and rp_2 are distinct. By (23) (applied to p_1, q_1, and r_1), $w(a) = w(pq_2) = w(qr_2) = w(rp_2)$.

By unique-cover-modularity, there is a p_3 covering rp_2 and pq_2, a q_3 covering pq_2 and qr_2, and an r_3 covering qr_3 and rp_3. Suppose two of p_3, q_3, and r_3 are the same, say p_3 is the same as q_3. Define $b = p_3 = q_3$, so b covers pq_2, qr_2, and rp_2. Then by (23) (applied to pq_2, qr_2, and rp_2), $w(b) = w(p_1) = w(q_1) = w(r_1)$. Since w is positive, (12) implies $2w(pq_2) \geq w(p_1) + w(q_1) = 2w(b)$. So $w(pq_2) \geq w(b)$ and similarly $w(qr_2) \geq w(b)$ and $w(rp_2) \geq w(b)$. Applying (12) to b and the above gives $2w(b) \geq w(pq_2) + w(qr_2) + w(rp_2) \geq 3w(b)$. Since $w(b) > 0$, this is a contradiction. Thus p_3, q_3, and r_3 are distinct.

By (23) (applied to pq_2, qr_2, and rp_2), $w(p_3) = w(p_1)$, $w(q_3) = w(q_1)$, and $w(r_3) = w(r_1)$. Applying (12) and positivity,

$$2w(p_1) + 2w(q_1) + 2w(r_1) = 2w(p_3) + 2w(q_3) + 2w(r_3)$$

$$2w(p_3) + 2w(q_3) + 2w(r_3) \geq 2w(pq_2) + 2w(qr_2) + 2w(rp_2) \tag{24}$$

$$2w(pq_2) + 2w(qr_2) + 2w(rp_2) \geq 2w(p_1) + 2w(q_1) + 2w(r_1) \tag{25}$$

Thus

$$w(p_1) + w(q_1) + w(r_1) = w(pq_2) + w(qr_2) + w(rp_2) = w(p_3) + w(q_3) + w(r_3)$$

and the inequalities (24) and (25) are equalities. Equation (12) and the criterion that w is positive requires that pq_2, qr_2, rp_2, p_3, q_3, and r_3 cover no elements other than the ones mentioned above (and shown in fig. 6) and have no upward orphans. Applying (12) to pq_2, qr_2, and rp_2 thus gives

$$w(a) = w(pq_2) = w(p_1)/2 + w(q_1)/2 \tag{26}$$

$$w(a) = w(qr_2) = w(q_1)/2 + w(r_1)/2 \tag{27}$$

$$w(a) = w(rp_2) = w(r_1)/2 + w(p_1)/2. \tag{28}$$

Together these prove $w(p_1) = w(q_1) = w(r_1) = w(a)$ and $w(p_3) = w(q_3) = w(r_3) = w(a)$.

By (13) and that w is positive, $2w(a) \geq w(p_1) + w(q_1) + w(r_1) = 3w(a)$. Since $w(a) > 0$, this is a contradiction.

Thus, there is no $a \in P$ that is covered by three distinct elements. □

Lemma 4.27. *No element of P covers three or more distinct elements.*

Proof.

This is proved dually to lem. 4.26, by interchanging the use of (12) and (13) and replacing "upward orphan" with "downward orphan". □

Remark 4.28. Lem. 4.26 and 4.27 are the deeper explanation of the long-known fact[Fom1994, sec. 2.2] that $\mathrm{Ideal_f}(\mathbb{N}^k)$ for $k > 2$ has no positive weighting as a differential lattice.

The equations implied by lem. 4.24 for any particular $p \in P$ are characterized by the structure of its neighborhood N_p as a poset.

Lemma 4.29. *N_p is characterized up to poset isomorphisms fixing p by the numbers $|C_p^-|$, $|C_p^+|$ and $|S_p|$: All elements of C_p^- are covered by p, all elements of C_p^+ cover p, and S_p forms a matching between subsets of C_p^- and*

C_p^+, with each element of S_p covering a distinct element of C_p^- and being covered by a distinct element of C_p^+. In addition, $|C_p^-| \le 2$, $|C_p^+| \le 2$, $|S_p| \le |C_p^+|$, $|S_p| \le |C_p^-|$, and $|N_p| = |C_p^-| + |C_p^+| + |S_p| + 1$.

Proof.
This follows from lem. 4.22, 4.23, 4.26, and 4.27. □

5. Global characterization

In this section, we continue by using the local characterization from sec. 4 to prove "global" facts about the poset P of points. These culminate in an enumeration of a small number of possible structures for P. All points $p \in P$ have a neighborhood N_p as described by lem. 4.29, and w must satisfy the consequent constraints per lem. 4.24.

5.1. P has a minimum element.

Lemma 5.1. *Given any two elements $a, b \in P$, we can connect them by a sequence of elements*

$$a = x_0, x_1, x_2, \ldots, x_i, \ldots, x_{n-1}, x_n = b$$

with

$$a = x_0 \gtrdot x_1 \gtrdot x_2 \gtrdot \cdots \gtrdot x_i \lessdot \cdots \lessdot x_{n-1} \lessdot x_n = b,$$

for some $0 \le i \le n$. (That is, the sequence first descends, then ascends.)

Proof.
Because P is locally finite and cannot be separated into the disjoint union of two non-empty posets, a and b can be connected with a sequence of elements $a = x_0, x_1, x_2, \ldots, x_n = b$ where each adjacent pair x_i, x_{i+1} has either $x_i \lessdot x_{i+1}$ or $x_i \gtrdot x_{i+1}$. Given such a sequence, if there is an i such that $0 < i < n$ and $x_{i-1} \lessdot x_i \gtrdot x_{i+1}$, that is, x_i is a local maximum, by the criterion of unique-cover-modularity, there is a y such that $x_{i-1} \gtrdot y \lessdot x_{i+1}$. Thus $a = x_0, x_1, x_2, \ldots, x_{i-1}, y, x_{i+1}, \ldots x_n = b$ is another such sequence.

We can iterate this transformation on the sequence. Each transformation preserves the number of adjacent pairs $x_i \gtrdot x_{i+1}$ and the number of adjacent pairs $x_i \lessdot x_{i+1}$, while moving the location of the $x_i \gtrdot x_{i+1}$ pairs earlier in the sequence. Thus any series of such transformations must terminate in a sequence for which there is no $i < i < n$ for which x_i which is a local maximum. Thus, any terminal sequence must have some $0 \le i \le n$ for which for all $0 \le j < i$, $x_j \gtrdot x_{j+1}$ and for all $i \le j < n$, $x_j \lessdot x_{j+1}$. □

Lemma 5.2. *P has a minimum element, which we denote $\hat{0}_P$.*

Proof.
Since P is non-empty and finitary by lem. 4.3(2), P must contain at least one minimal element m. Assume there is an element $a \not\ge m$ in P. Then by lem. 5.1, there is a sequence of elements $m = x_0, x_1, x_2, \ldots, x_n = a$ where for some $0 \le i \le n$, for any $0 \le j < i$, $x_j \gtrdot x_{j+1}$ and for any $i \le j < n$, $x_j \lessdot x_{j+1}$. But since m is minimal, there is no y with $m \gtrdot y$, and so $i = 0$. If $n > 0$, then $m = x_0 \lessdot x_1 \le x_n = a$. That implies $m < a$, which contradicts that $a \not\ge m$. So $n = 0$ and $m = x_0 = a$, which also contradicts $a \not\ge m$. Thus there is no $a \not\ge m$ in P and m is the minimum of P. □

Lemma 5.3. *P is graded. We define ρ as the rank function on P. $\rho(\hat{0}_P) = 0$.*

Proof.
Given that P is locally finite, is unique-cover-modular, and has a minimum element $\hat{0}_P$, a straightforward generalization of [Birk1967, §II.8 Th. 14] proves the conclusion. □

Remark 5.4. *In lem. 5.3, the hypothesis that $\hat{0}_P$ exists is necessary. A counterexample is the poset "3-Plait" in* [Fom1994, Fig. 7 and Exam. 2.2.10]*. Further counterexamples are the posets of points of the lattices of cylindrical partitions.*[El2025]

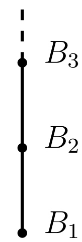


FIGURE 7. The case $N = \infty$ where the $B_\bullet$ form an upward semi-infinite path.

5.2. The bottom elements of P. First we define the sequence, finite or infinite, of elements at the bottom of P that form a chain, with each element (above $\hat{0}_P$) being the unique cover of the one below.

Definition 5.5. *We define $B_1 = \hat{0}_P$. Then iteratively for each $i \geq 0$, if B_i has a unique upward cover, we define B_{i+1} to be that cover. If there is a $B_\bullet$ that does not have a unique cover (and thus by lem. 4.26 has either zero or two covers), we define N to be the index of that $B_\bullet$. Otherwise, we define N to be ∞. (B for "bottom".)*

In the case $N = \infty$, the $B_\bullet$ form an upward semi-infinite path, and they are the entirety of P. The constraints on w are $w(B_1) = r$, $2w(B_1) = w(B_2)$, and, for all $i \geq 2$, $2w(B_i) = w(B_{i-1}) + w(B_{i+1})$, which have the unique solution $w(B_i) = ir$. Thus we have:

Case 5.6. *(sec. 2.2 item (4)) P is the upward semi-infinite path B_i for $i \geq 1$ with $r > 0$ and $w(B_i) = ir$. L is isomorphic to $\mathbb{N}$.*

If N is not ∞, then there are two cases: B_N has zero upward covers and B_N has two upward covers. If B_N has zero upward covers, then P consists of the finite path $B_0 \lessdot B_1 \lessdot B_2 \lessdot \cdots \lessdot B_N$. By lem. 4.24, the constraints on w

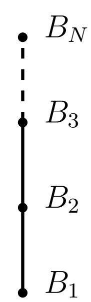


FIGURE 8. The case N is not ∞, B_N has zero upward covers, and the $B_\bullet$ form a finite path.

are

- $w(B_1) = r$;
- if $N = 1$, $2w(B_1) = 0$;
- if $N \geq 2$, $2w(B_1) = w(B_2)$;
- for all $2 \leq i < N$, $2w(B_i) = w(B_{i-1}) + w(B_{i+1})$; and
- if $N \geq 2$, $2w(B_N) = w(B_{N-1})$.

For all finite N, these imply $r = 0$ and all $w(B_i) = 0$, so w is not positive.

5.3. P^+ is two-dimensional. The remaining case is that B_N has exactly two upward covers.

Definition 5.7. *We define P^+ as the elements of P that are not a B_i for which $i < N$.*

Lemma 5.8. *P^+ has at least three elements: one is B_N and two are the covers of B_N. All $p \in P^+$ are $\geq B_N$, and each has a finite grade $\rho(p) \geq \rho(B_N)$ by lem. 5.3.*

We will "draw the Hasse diagram of P^+ in the plane" by defining a set of locations T_{xy} for all integers $x \geq 0$ and $y \geq 0$. As a poset, $T \cong \mathbb{N} \times \mathbb{N}$. Then we assign every $p \in P^+$ to a distinct location T_{xy} with $x + y = \rho(p) - \rho(B_N) = \rho(p) - N$. ($T$ for "top elements".) That is, B_N is assigned to T_{00} and elements of P^+ are assigned "upward" from T_{00}. If $p \lessdot q$ in P^+ and p is assigned to T_{xy} then q is assigned to either $T_{x+1,y}$ or $T_{x,y+1}$. Conversely, if both T_{xy} and either $T_{x+1,y}$ or $T_{x,y+1}$ are assigned, the latter covers the former. Intuitively, the $T_{\bullet\bullet}$ form a quarter-plane extending upward, with T_{00} at the bottom center, the x coordinate increasing to the upper-left and the y coordinate increasing to the upper-right. Note that all elements of P^+ are assigned to distinct locations $T_{\bullet\bullet}$, but some locations may not have an assigned element. We abuse notation by using T_{xy} to denote both the location and, if an element of P^+ is assigned to that location, that element of P^+.

We define the coordinate $v = x + y$; v is how far T_{xy} is vertically above T_{00}. We define the coordinate $h = y - x$; h is how far T_{xy} is horizontally left of T_{00} (if negative) or right of T_{00} (if positive). Note that while v ranges over all nonnegative integers, h ranges over integers $-v \leq h \leq v$ which have the same parity as v. The assignment of elements of P^+ realizes its Hasse diagram, with each cover relation being either one step to the upper left (incrementing x) or one step to the upper right (incrementing y). An example assignment is shown in fig. 9.

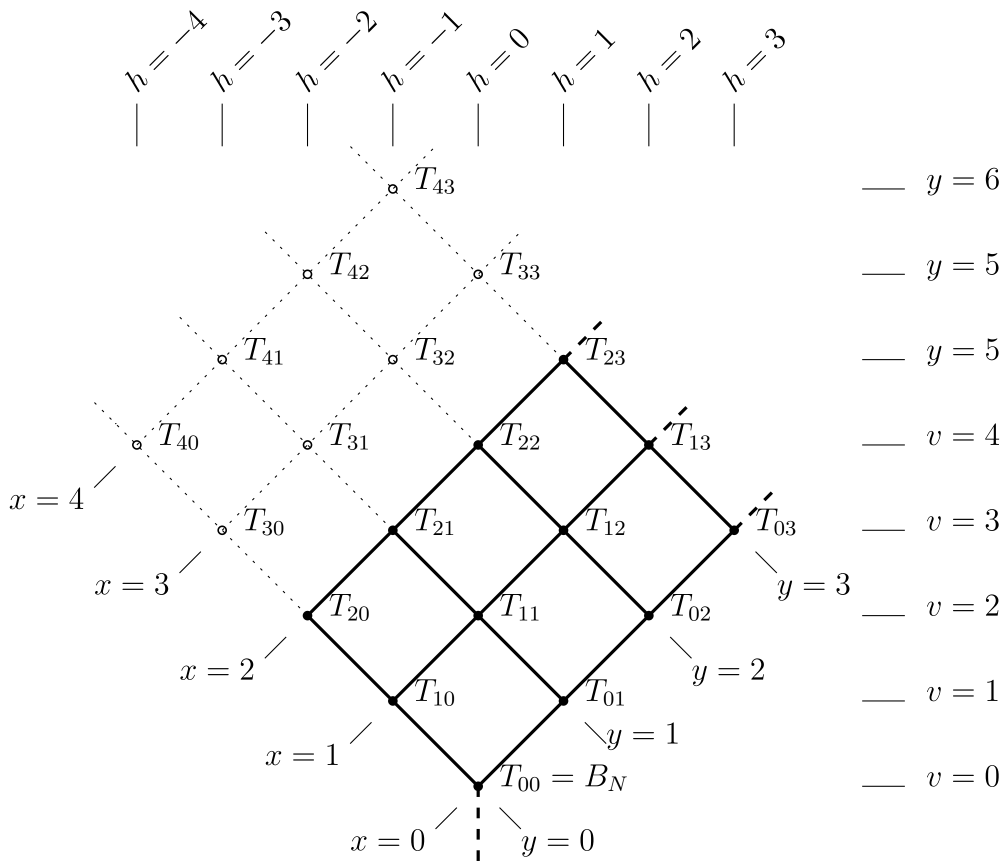


FIGURE 9. An example of assigning elements of P^+ to locations $T_{\bullet\bullet}$. Solid circles are assigned locations, empty circles are unassigned locations. Solid lines are coverings in P. Further elements are assigned to the upper-right of T_{03}, T_{13}, and T_{23}. B_i for $i < N$ are assigned downward from T_{00}.

We construct the assignments by first assigning $T_{00} = B_N$ and successively assigning each grade of P^+ above N to locations so as to satisfy these rules.

Lemma 5.9. *We can assign the elements of P^+ to locations in $\{T_{xy} \mid x \geq 0 \text{ and } y \geq 0\}$ by the following process, which has the following properties:*

(1) Assign B_N to T_{00}.

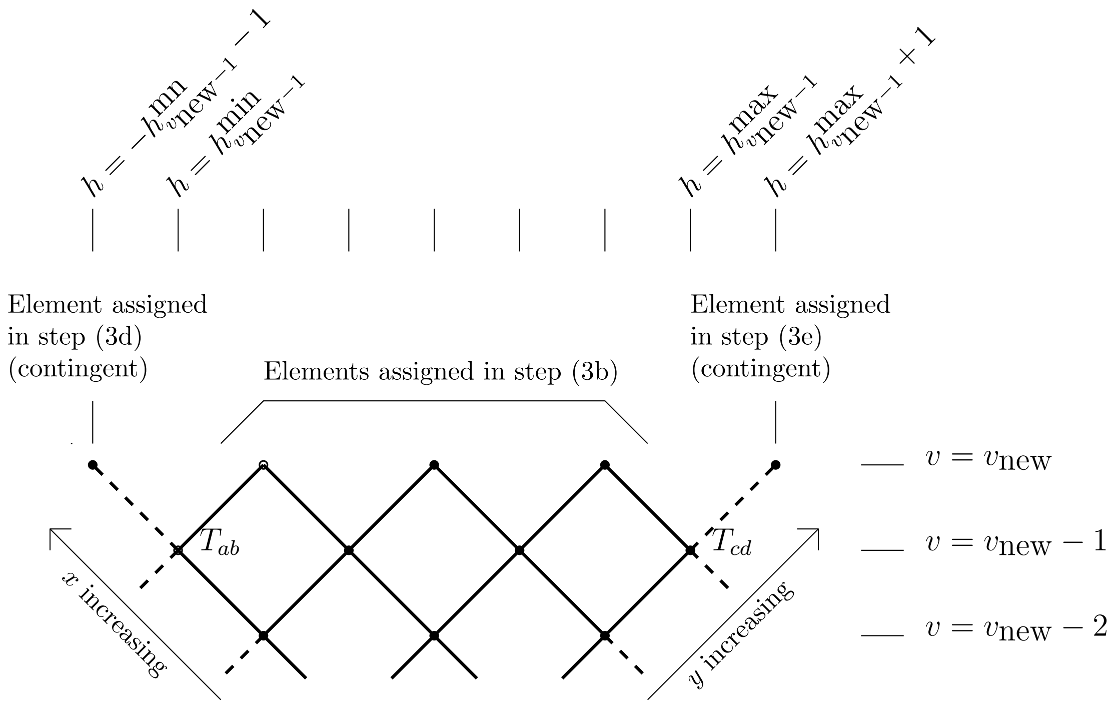


FIGURE 10. An example of assigning elements of P^+ with rank $v_{\mathrm{new}} + R$ to locations with $v = v_{\mathrm{new}}$ in steps (3a–f). T_{ab} has $v = v_{\mathrm{new}} - 1$ and $h = h^{\min}_{v_{\mathrm{new}}-1}$ and T_{cd} has $v = v_{\mathrm{new}} - 1$ and $h = h^{\max}_{v_{\mathrm{new}}-1}$.

(2) Assign rank $N-1$, the two covers of B_N, arbitrarily to T_{10} and T_{01}.

(3) Iteratively consider the grades of P^+ from $N+2$ upward. Terminate if we reach a grade that is empty (all elements of P^+ have been assigned):

(a) Within grade R, we will assign the elements of that grade to distinct locations T_{xy} with $x+y = v_{\mathrm{new}} = R-N$. At this point, these iteration hypotheses are true of the T_{xy} with $x+y < v_{\mathrm{new}}$:

(i) The elements assigned to locations with a particular value of $v < v_{\mathrm{new}}$ have values h that are a contiguous subset of the integers $-v \le h \le v$ which have the same parity as v. Define $h^{\min}_v$ to be the minimum h of an assigned location with a particular value of v and define $h^{\max}_v$ to be the maximum h of an assigned location with a particular value of v. Similarly, define $x^{\min}_v$, $x^{\max}_v$, $y^{\min}_v$, and $y^{\max}_v$ as the minimum and maximum x and y coordinates of assigned locations with this value of v.

(ii) For any $0 < v < v_{\mathrm{new}}$, $h^{\min}_v = h^{\min}_{v-1} \pm 1$ and $h^{\max}_v = h^{\max}_{v-1} \pm 1$. Thus $x^{\min}_v = x^{\min}_{v-1}$ or $x^{\min}_{v-1} + 1$, $x^{\max}_v = x^{\max}_{v-1}$ or $x^{\max}_{v-1} + 1$, $y^{\min}_v = y^{\min}_{v-1}$ or $y^{\min}_{v-1} + 1$, and $y^{\max}_v = y^{\max}_{v-1}$ or $y^{\max}_{v-1} + 1$,

(iii) If $p \lessdot q$ in P^+ are assigned at this point, then q is assigned to some T_{xy} and p is assigned to either $T_{x-1,y}$ or $T_{x,y-1}$.

(iv) Conversely, if T_{xy} is assigned and either of $T_{x-1,y}$ or $T_{x,y-1}$ is assigned, then the former covers the latter in P^+.

(b) Consider any adjacent pair of assigned locations with $v = v_{\mathrm{new}} - 1$, T_{xy} and $T_{x-1,y+1}$, with $x+y = v_{\mathrm{new}} - 1$, $h^{\min}_{v_{\mathrm{new}}-1} \le y-x$, and $(y+1)-(x-1) \le h^{\max}_{v_{\mathrm{new}}-1}$. T_{xy} and $T_{x-1,y+1}$ cover $T_{xy} \wedge T_{x-1,y+1} = T_{x-1,y}$ and so they must be covered by $T_{xy} \vee T_{x-1,y+1}$, which has rank R in P^+. Assign $T_{xy} \vee T_{x-1,y+1}$ to location $T_{x,y+1}$. By lem. 4.27, $T_{xy} \vee T_{x-1,y+1}$ covers no element other than T_{xy} and $T_{x-1,y+1}$.

(c) Every element of rank R not yet assigned necessarily covers at least one element of rank $R-1$. But any element of rank $R-1$ with $h^{\min}_{v_{\mathrm{new}}-1} < h < h^{\max}_{v_{\mathrm{new}}-1}$ already has two elements that cover it that were assigned in step (3b) that cover it, so the only elements of rank $R-1$ that it might cover are

(i) the assigned location with minimum h, $T_{ab} = T_{x^{\max}_{v_{\mathrm{new}-1}}, y^{\min}_{v_{\mathrm{new}-1}}}$ and/or

(ii) the assigned location with maximum h, $T_{cd} = T_{x^{\min}_{v_{\mathrm{new}-1}}, y^{\max}_{v_{\mathrm{new}-1}}}$

(which might be the same location). However, it cannot cover both of these elements if they are distinct. If it did, then both of these elements would cover an element of rank $R-2$, which would have caused this element (of rank R) to have been assigned a location in step (3b).

(d) *If an unassigned element covers T_{ab}, but T_{ab} has another assigned cover (which must be $T_{a,b+1}$), assign this element to $T_{a+1,b}$. There cannot be two such elements in a single rank, as then T_{ab} would be covered by three elements.*

(e) *If an unassigned element covers T_{pq}, but T_{pq} has another assigned cover (which must be $T_{p+1,q}$), assign this element to $T_{p,q+1}$. There cannot be two such elements in a single rank, as then T_{pq} would be covered by three elements.*

(f) *The remaining case is $T_{ab} = T_{pq}$ and unassigned elements remain. By lem. 4.26, there are at most two such elements, which can be assigned arbitrarily to $T_{a+1,b}$ and $T_{a,b+1}$. (Ultimately, we will prove that this only happens in rank $N+1$, which was handled by step (2).)*

(g) *At this point, all elements in rank R of P^+ are assigned locations, and the iteration hypotheses (3(a)i)–(3(a)iv) are true of the T_{xy} with $x+y \leq v_{\text{new}}$.*

The algorithm is nondeterministic in cases (2) and (3f); either cover of the unique element in one rank may be assigned to either of the two locations in the next rank. In all other steps, it is deterministic. The typical operation of steps (3b–f) is shown in fig. 10.

Proof.
The induction properties stated in the lemma and the correctness of the algorithm are proved by mutual induction. The details are tedious but straightforward. □

Because the assigned locations of each grade are contiguous and very similar to the assigned locations of the grades above and below, the possible P^+ are determined up to isomorphism by the outline of the set of assigned locations, the values $h_\bullet^{\min}$ and $h_\bullet^{\max}$.

5.4. The bottom of P^+. We can now examine the constraints implied by the structure of the neighborhoods of P.

Definition 5.10. *We define $w_{xy} = w(T_{xy})$ when T_{xy} is allocated, $m = w(B_N) = w_{00}$, $\delta^- = w_{10} - m$, and $\delta^+ = w_{01} - m$. If $N > 1$, we define $\epsilon = w_{B_{N-1}} - m$ and otherwise $\epsilon = -m$.*

(m for the "weight of the *middle* element of P^+", namely $T_{00} = B_N$; δ^- for "the *difference* in the *negative* direction of h", that is, toward the left side of P^+; δ^+ for "the *difference* in the *positive* direction of h", that is, toward the right side of P^+.) Both δ^- and δ^+ are well-defined since $T_{00} = B_N$ is assumed to have two upward covers, which are T_{10} and T_{01}.

Lemma 5.11. *The constraints on w due to applying lem. 4.24 to B_1 and all the B_i for $i < N$ are*

$$w(B_i) = ir \qquad \text{for all } 1 \leq i \leq N \tag{29}$$

In particular, $m = Nr$, $\epsilon = -r < 0$, and $N = -m/\epsilon$.

Lemma 5.12. *The constraints on w due to applying lem. 4.24 to $T_{\bullet 0}$ imply either*

(1) $\delta^- \geq 0$:*we define $x_L = \infty$, all $T_{\bullet 0}$ are allocated, and $w_{i0} = m + i\delta^-$ for all $0 \leq i$; or*

(2) $\delta^- < 0$:*we define $x_L = -m/\delta^-$, x_L is a positive integer ≥ 2, T_{i0} is allocated iff $i < x_L$, and $w_{i0} = m + i\delta^-$ for all $0 \leq i < x_L$.*

Proof.
This is proved by finding the smallest I for which T_{i0} is not allocated, if one exists. Because T_{01} is allocated, unique-modular-covering guarantees that for any T_{i0} with $i \leq I$ (any i if I does not exist), T_{i1} is allocated and is not an upward orphan. The constraints on w_{i0} are that $2w_{i0} = w_{i-1,0} + w_{i+1,0}$ for any $0 < i < I$ and $2w_{i0} = w_{i-1,0}$ for $i = I$ (if I exists). Thus, $w_{i0} = m + i\delta^-$ for any $0 \leq i < I$ (and for any $0 \leq i$ if I does not exist). If no such I exists, conclusion case (1) follows, since positivity for all of the $w_{\bullet 0}$ requires $\delta^- \geq 0$. If I exists, conclusion case (2) follows, since the constraint on $w_{I-1,0}$ requires it to equal $-\delta^-$. □

Lemma 5.13. *Similarly, the constraints on w due to applying lem. 4.24 to $T_{0\bullet}$ imply either*

(1) $\delta^+ \geq 0$:*we define $y_L = \infty$, all $T_{0\bullet}$ are allocated, and $w_{0i} = m + i\delta^+$ for all $0 \leq i$; or*

(2) $\delta^+ < 0$:*we define* $y_L = -m/\delta^+$, y_L *is a positive integer* ≥ 2, T_{0i} *is allocated iff* $i < y_L$, *and* $w_{0i} = m + i\delta^+$ *for all* $0 \leq i < y_L$.

Definition 5.14. *We define* $\Delta^- = -m/\delta^-$ *and* $\Delta^+ = -m/\delta^+$, *defining* $-m/0 = \infty$. *This parallels the relationship between* N *and* ϵ, $N = -m/\epsilon$.

Lemma 5.15. Δ^- *is either* ∞ *(if* $\delta^- = 0$*), a negative real number (if* $\delta^- > 0$*), or the reciprocal of a positive integer (which is* x_L, *if* $\delta^- < 0$*). Similarly for* Δ^+ *w.r.t.* y_L.

Lemma 5.16. *The constraint on* T_{00} *is equivalent to*

$$\frac{1}{\Delta^-} + \frac{1}{\Delta^+} + \frac{1}{N} = 1, \tag{30}$$

defining $1/\infty = 0$.

Proof. Applying lem. 4.24 to T_{00} yields $2w_{00} = (w_{00} + \epsilon) + w_{10} + w_{01}$, since T_{10} and T_{01} both exist and are upward orphans of T_{00}. Dividing this equation by $-m$ gives the conclusion. □

Lemma 5.17. *Not considering the order of the terms, simple enumeration shows the decompositions of* 1 *as a sum of reciprocals as in (30) are:*

$$\frac{1}{1} + \frac{1}{n} + (-\frac{1}{n}) \quad \text{for any integer } n \geq 1$$
$$\frac{1}{1} + \frac{1}{\infty} + \frac{1}{\infty}$$
$$\frac{1}{2} + \frac{1}{2} + \frac{1}{\infty}$$
$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{4}$$
$$\frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

5.5. The possible cases for P.

Lemma 5.18. *Since*

(1) *the possible decompositions of 1 as a sum of reciprocals are listed in lem. 5.17,*
(2) N *must be positive,*
(3) *given the left-right symmetry of* P, *we conventionally choose* $x_L \geq y_L$, *and*
(4) T_{10} *and* T_{01} *are allocated, so* $\Delta^- \neq 1$ *and* $\Delta^+ \neq 1$,

the possible cases for P *are limited to:*

Δ^-	x_L	Δ^+	y_L	N	Lattice
$-n$ for $n \geq 2$	∞	n	n	1	$\mathbb{Y}_n$
∞	∞	∞	∞	1	$\mathbb{Y}$
∞	∞	2	2	2	$\mathbb{SY}$
3	3	2	2	6	none (lem. 5.21)
6	6	2	2	3	none (lem. 5.23)
6	6	3	3	2	none (lem. 5.20)
4	4	2	2	4	none (lem. 5.22)
4	4	4	4	2	none (lem. 5.20)
3	3	3	3	3	none (lem. 5.20)

Remark 5.19. *If we do not require $\hat{0}_P$ to exist, then we remove the restriction $N > 0$ in lem. 5.18. That allows the additional case $\Delta^- = \Delta^+ = 2$, and $N = \infty$.*

Lemma 5.20. *It is not possible that x_L and y_L are not ∞ and are > 2.*

Proof.
If x_L and y_L are not ∞ and are > 2, then T_{20} and T_{02} are allocated and neither T_{21} nor T_{12} are upward orphans of T_{11}. Thus the constraints of lem. 4.24 for T_{11} simplifies to $0 = \delta^- + \delta^+$. But since x_L and y_L are finite, $\delta^- < 0$ and $\delta^+ < 0$, which is a contradiction. □

Lemma 5.21. *It is not possible that $x_L = 3$ and $y_L = 2$.*

Proof.

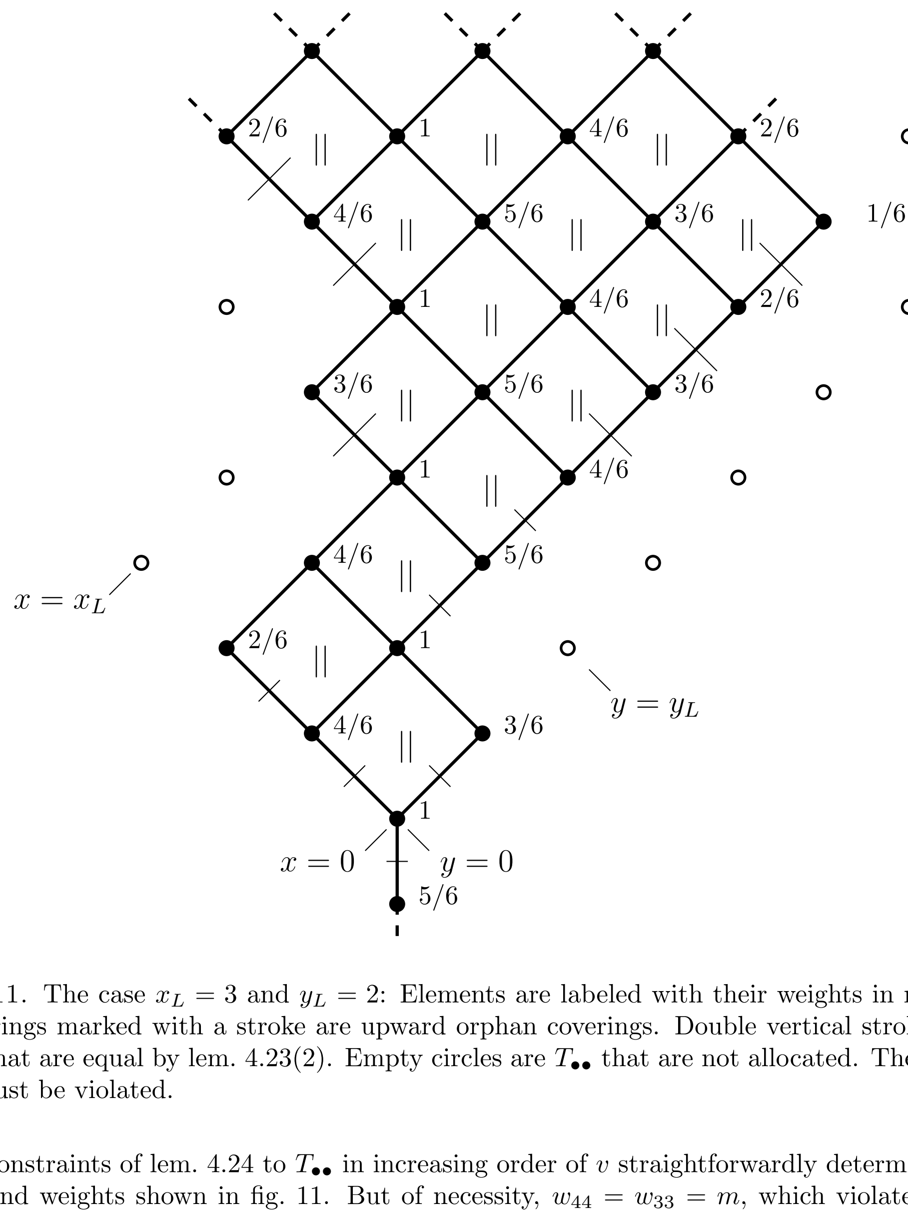


FIGURE 11. The case $x_L = 3$ and $y_L = 2$: Elements are labeled with their weights in multiples of m. Coverings marked with a stroke are upward orphan coverings. Double vertical strokes connect weights that are equal by lem. 4.23(2). Empty circles are $T_{\bullet\bullet}$ that are not allocated. The constraint on T_{44} must be violated.

Applying the constraints of lem. 4.24 to $T_{\bullet\bullet}$ in increasing order of v straightforwardly determines the allocations, non-allocations, and weights shown in fig. 11. But of necessity, $w_{44} = w_{33} = m$, which violates the constraint on T_{44}. □

Lemma 5.22. *It is not possible that $x_L = 4$ and $y_L = 2$.*

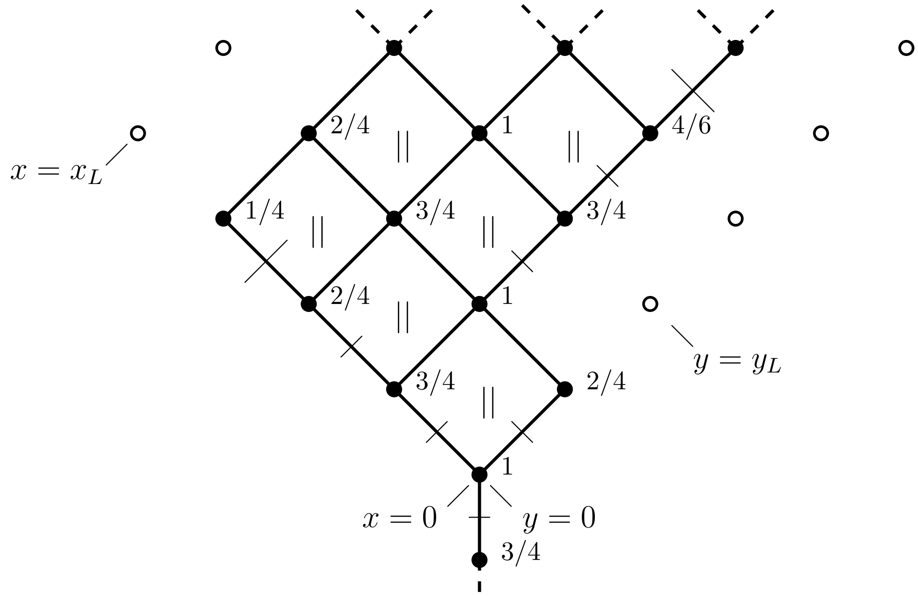


FIGURE 12. The case $x_L = 4$ and $y_L = 2$: Elements are labeled with their weights in multiples of m. Coverings marked with a stroke are upward orphan coverings. Double vertical strokes connect weights that are equal by lem. 4.23(2). Empty circles are $T_{\bullet\bullet}$ that are not allocated. The constraint on T_{22} must be violated.

Proof.

Applying the constraints of lem. 4.24 to $T_{\bullet\bullet}$ in increasing order of v straightforwardly determines the allocations, non-allocations, and weights shown in fig. 12. But of necessity, $w_{22} = w_{11} = m$, which violates the constraint on T_{22}. □

Lemma 5.23. *It is not possible that $x_L = 6$ and $y_L = 2$.*

Proof.

Applying the constraints of lem. 4.24 to $T_{\bullet\bullet}$ in increasing order of v straightforwardly determines the allocations, non-allocations, and weights shown in fig. 13. But of necessity, $w_{22} = w_{11} = m$, which violates the constraint on T_{22}. □

Lemma 5.24. *For the first three cases of lem. 5.18, L must be:*

(1) for $\Delta^- = -n$ and $\Delta^+ = n$, L is isomorphic to $\mathbb{Y}_n$ with weights that are m/n times the canonical weights (sec. 2.2(3) and fig. 3),
(2) for $\Delta^- = \Delta^+ = \infty$, L is isomorphic to $\mathbb{Y}$ with weights that are m times the canonical weights (sec. 2.2(1) and fig. 1),
(3) for $\Delta^- = \infty$ and $\Delta^+ = 2$, L is isomorphic to $\mathbb{SY}$ with weights that are $m/2$ times the canonical weights (sec. 2.2(2) and fig. 2),

with P being the poset of join-irreducibles of L with the corresponding weights.

Proof.

In all of these cases, the values w_{B_i}, w_{x0}, and w_{0y} are determined (relative to m) by x_L, y_L, and N. Which other elements of $T_{\bullet\bullet}$ are allocated and their weights are straightforwardly determined by the constraints of lem. 4.24 on them (working in order of increasing v), and necessarily result in the known lattices listed in the table. □

6. Classification theorem

Theorem 6.1. *The only Fomin lattices L that satisfy the criteria*

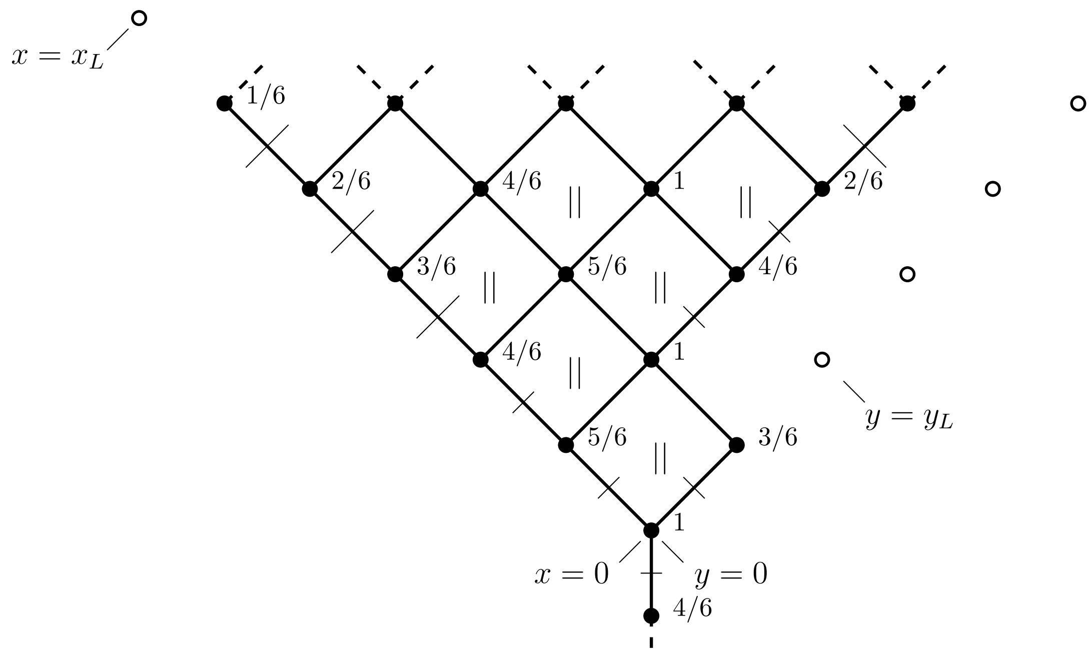


FIGURE 13. The case $x_L = 6$ and $y_L = 2$: Elements are labeled with their weights in multiples of m. Coverings marked with a stroke are upward orphan coverings. Double vertical strokes connect weights that are equal by lem. 4.23(2). Empty circles are $T_{\bullet\bullet}$ that are not allocated. The constraint on T_{22} must be violated.

(1) The lattice L is distributive with $\hat{0}_L$. We define the poset of points (join-irreducibles) to be P, so $L = \mathrm{Ideal_f}(P)$.
(2) P is unique-cover-modular.
(3) The weighting w is positive, that is, the set of values V, the target of w, is $\mathbb{Z}$, and all of the values of w are positive. The differential degree r is positive.
(4) The lattice cannot be factored into two nontrivial lattices, which is equivalent to assuming that P cannot be partitioned into two disjoint non-empty posets $P = P_1 \sqcup P_2$.

are the cases

$\mathbb{Y}$: *(sec. 2.2(1)) P is isomorphic to the quadrant of the plane,*
$$\{(x,y) \,|\, x \text{ and } y \text{ are integers} \geq 0\},$$
and L is isomorphic to $\mathbb{Y}$, Young's lattice, the lattice of partitions, with weighting r times the canonical weighting of 1.

$\mathbb{SY}$: *(sec. 2.2(2)) P is isomorphic to the octant of the plane,*
$$\{(x,y) \,|\, x \text{ and } y \text{ are integers with } x \geq 0 \text{ and } 0 \leq y \leq x\},$$
and L is isomorphic to $\mathbb{SY}$, the shifted Young's lattice, the lattice of partitions into distinct parts, with weighting r times the canonical weighting of 1 and 2.

$\mathbb{Y}_\bullet$: *(sec. 2.2(3)) P is isomorphic to the k-row strip for some $k \geq 1$,*
$$\{(x,y) \,|\, x \text{ and } y \text{ are integers, } 0 \leq x \text{ and } 0 \leq y < k\},$$
and L is isomorphic to $\mathbb{Y}_k$, the k-row Young's lattice, the lattice of partitions with at most k parts, with weighting r/k times the canonical weighting $w(T_{xy}) = k + x - y$.

Proof.

Note that conclusion case $\mathbb{Y}_k$ comprises case 5.6, the upward semi-infinite path, as $\mathbb{Y}_1$ and case lem. 5.24(1) as $\mathbb{Y}_k$ with $k \geq 2$. The theorem is the summation of case 5.6, lem. 5.18, 5.20, 5.21, 5.22, 5.23, and 5.24. □

7. Future directions

The strength of the criteria of our classification theorem (th. 6.1) is disappointing, but they are satisfied by all known Fomin lattices except the one family that is not even distributive, the Young–Fibonacci lattices and cartesian products that include them. In future work, we hope to expand the classification to larger classes.

Relaxing the criterion that poset of points is unique-cover-modular seems like it may be tractable. The structure of neighborhoods would become more complicated but perhaps a careful analysis of the graphs of possible coverings would show that a tractable set of constraints characterize the weights of the points.

On the other hand, the technique used in [Stan2012, Exer. 3.51(a) soln.] can be restated to provide a way to analyze an arbitrary differential distributive lattice, or conversely, to provide a nondeterministic algorithm for constructing all differential distributive lattices for a wide range of variations of "differential". This can be used to prove that no point is covered by three distinct points (lem. 4.26) and that there is a unique minimum point (lem. 5.2). It may be possible to use this technique to prove additional lemmas used in this analysis without assuming unique-cover-modularity.

Classifying modular but non-distributive Fomin lattices seems to be the most difficult problem. There seems to be no tractable representation of general modular lattices that parallels Birkhoff's Representation Theorem for distributive lattices, and our entire analysis is based on characterizing L in the manner of Birkhoff's representation. Jonathan Farley notes that the analysis of [FagHerr1981], especially Cor. 4.8, the "Verbindungssatz", may provide a basis for representing general modular lattices.

Richard Stanley asks how much of this analysis can be applied to sequentially differential posets[Stan1990, sec. 2][5] that are distributive lattices.

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Email address: worley@alum.mit.edu