Contents

Solving Problems in Special Relativity with Quaternions

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1 Doing the work of the Lorentz group with quaternion rotations and dilations

Introduction

In 1905, Einstein proposed the principles of special relativity without a deep knowledge of the mathematical structure behind the work. He had to rely on his old math teacher Minkowski to learn the theory of transformations (I do not know the details of Einstein's education, but it could make an interesting discussion :-) Eventually, Einstein understood general transformations, embodied in the work of Riemann, well enough to formulate general relativity.

A. W. Conway and L. Silberstein proposed a different mathematical structure behind special relativity in 1911 and 1912 respectively (a copy of Silberstein's work is on the web. Henry Baker has made it available at ftp://ftp.netcom.com/-pub/hb/hbaker/quaternions/). Cayley had observed back in 1854 that rotations in 3D could be achieved using a pair of quaternions with a norm of one:

$$q' = a\,q\,b$$

$$\text{where } a^* a = b^* b = 1$$

If this works in 3D space, why not do the 4D transformations of special relativity? It turns out that a and b must be complex-valued quaternions, or biquaternions. Is this so bad? Let me quote P.A.M. Dirac (Proc. Royal Irish Academy A, 1945, 50, p. 261):

"Quaternions themselves occupy a unique place in mathematics in that they are the most general quantities that satisfy the division

axiom–that the product of two factors cannot vanish without either factor vanishing. Biquaternions do not satisfy this axiom, and do not have any fundamental property which distinguishes them from other hyper-complex numbers. Also, they have eight components, which is rather too many for a simple scheme for describing quantities in space-time."

Just for the record: plenty of fine work has been done with biquaternions, and I do not deny the validity of any of it. Much effort has been directed toward "other hyper-complex numbers", such as Clifford algebras. For the record, I am making a choice to focus on quaternions for reasons outlined by Dirac.

Dirac took a Mobius transformation from complex analysis and tried to develop a quaternion analog. The approach is too general, and must be restricted to graft the results to the Lorentz group. I personally have found this approach hard to follow, and have yet to build a working model of it in Mathematica. I needed something simpler :-)

Rotation + Dilation

Multiplication of complex numbers can be thought of as a rotation and a dilation. Conway and Silberstein's proposals only have the rotation component. An additional dilation term might allow quaternions to do the necessary work.

C. Möller wrote a general form for a Lorentz transformation using vectors ("The Theory of Relativity", QC6 F521, 1952, eq. 25). For fixed collinear coordinate systems:

$$\vec{X}' = \vec{X} + (\gamma - 1)\left(\vec{V}.\vec{X}\right)\frac{\vec{V}}{|\vec{V}|^2} - \gamma\, t\, \vec{V}$$

$$t' = \gamma\, t - \gamma\left(\vec{V}.\vec{X}\right)$$

$$\text{where } c = 1,\ \gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$$

If V is only in the i direction, then

$$\vec{X}' = \left(\gamma\vec{X} - \gamma t \vec{V}\right)\hat{i} + y\hat{j} + z\hat{k}$$

$$t' = \gamma t - \gamma\left(\vec{V}.\vec{X}\right)$$

The additional complication to the X' equation handles velocities in different directions than i.

This has a vector equation and a scalar equation. A quaternion equation that would generate these terms must be devoid of any terms involving cross products. The symmetric product (anti-commutator) lacks the cross product;

$$\{q, q'\} = \frac{q q' + q' q}{2} = \left(t t' - \vec{X}.\vec{X}', t\vec{X} + \vec{X} t'\right)$$

Möller's equation looks like it should involve two terms, one of the form AqA (a rotation), the other Bq (a dilation).

$$q' = q + (\gamma - 1)\frac{\left\{\left\{\left(\vec{V}\right)^*, q\right\}, \vec{V}\right\}}{|\vec{V}|^2} + \gamma\left\{\left(\vec{V}\right)^*, q^*\right\}$$

$$= q + (\gamma - 1)\frac{\left\{\left(\vec{V}.\vec{X}, -t V\right), \left(0, \vec{V}\right)\right\}}{|\vec{V}|^2} + \gamma\left\{\left(0, -\vec{V}\right), \left(t, -\vec{X}\right)\right\}$$

$$= \left(t, \vec{X}\right) + (\gamma - 1)\left(t, \left(\vec{V}.\vec{X}\right)\frac{\vec{V}}{|\vec{V}|^2}\right) - \gamma\left(\left(\vec{V}.\vec{X}\right), t\vec{V}\right)$$

This is the general form of the Lorentz transformation presented by Möller. Real quaternions are used in a rotation and a dialation to perform the work of the Lorentz group.

Implications

Is this result at all interesting? A straight rewrite of Moller's equation would have been dull. What is interesting is the equation which generates the Lorentz transformation. Notice how the Lorentz transformation depends linearly on q, but the generator depends on q and q*. That may have interesting interpretations. The generator involves only symmetric products. There has been some question in the literature about whether special relativity handles rotations correctly. This is probably one of the more confusing topics in physics, so I will just let the observation stand by itself.

Two ways exist to use quaternions to do Lorentz transformations (to be discussed in the next web page). The other technique relies on the property of a division algebra. There exists a quaternion L such that:

$$q' = L q$$

$$\text{such that } \operatorname{scalar}(q', q') = \operatorname{scalar}(q, q) = t^2 - \vec{X}.\vec{X}$$

For a boost along the i direction,

$$L = \frac{q'}{q} = \frac{(\gamma t - \gamma v x, -\gamma v t + \gamma x, y, z)(t, -x, -y, -z)}{(t^2 + x^2 + y^2 + z^2)}$$

$$= (\gamma t^2 - 2\gamma t v x + \gamma x^2 + y^2 + z^2, \gamma v(-t^2 + x^2),$$
$$t y - x z - \gamma t(y + v z) + \gamma x(v y + z),$$
$$t z + xy + \gamma t(v y - z) + \gamma x(-y + v z))\Big/$$
$$(t^2 + x^2 + y^2 + z^2)$$

$$\text{if } x = y = z = 0, \text{ then } L = (\gamma, -\gamma v, 0, 0)$$

$$\text{if } t = y = z = 0, \text{ then } L = (\gamma, \gamma v, 0, 0)$$

The quaternion L depends on the velocity and can depend on location in spacetime (85% of the type of problems assigned undergraduates in special relativity use an L that does not depend on location in spacetime). Some people view that as a bug, but I see it as a modern feature found in the standard model and general relativity as the demand that all symmetry is local. The existence of two approaches may be of interest in itself.

2 An alternative algebra for Lorentz boosts

Introduction

Many problems in physics are expressed efficiently as differential equations whose solutions are dictated by calculus. The foundations of calculus were shown in turn to rely on the properties of fields (the mathematical variety, not the ones in physics). According to the theorem of Frobenius, there are only three finite dimensional fields: the real numbers (1D), the complex numbers (2D), and the quaternions (4D). Special relativity stresses the importance of 4-dimensional Minkowski spaces: spacetime, energy-momentum, and the electromagnetic potential. In this notebook, events in spacetime will be treated as the 4-dimensional field of quaternions. It will be shown that problems involving boosts along an axis of a reference frame can be solved with this approach.

The tools of special relativity

Three mathematical tools are required to solve problems that arise in special relativity. Events are represented as 4-vectors, which can be add or subtracted, or multiplied by a scalar. To form an inner product between two vectors requires the Minkowski metric, which can be represented by the following matrix (where c = 1).

$$M_{metric} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};$$

$$\{t, x, y, z\}.M_{metric}.\{t, x, y, z\}$$

$$t^2 - x^2 - y^2 - z^2$$

$$g_\mu{}^\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\{t, x, y, z\}.g_\mu{}^\nu.\{t, x, y, z\} = t^2 - x^2 - y^2 - z^2$$

The Lorentz group is defined as the set of matrices that preserves the inner product of two 4-vectors. A member of this group is for boosts along the x axis, which can be easily defined.

$$\gamma[\beta\_] := \frac{1}{\sqrt{1 - \beta^2}}$$

$$\Lambda_x[\beta\_] := \begin{pmatrix} \gamma[\beta] & -\beta\gamma[\beta] & 0 & 0 \\ -\beta\gamma[\beta] & \gamma[\beta] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\gamma[\beta\_] := \frac{1}{\sqrt{1 - \beta^2}}$$

$$\Lambda_x[\beta\_] := \begin{pmatrix} \gamma[\beta] & -\beta\gamma[\beta] & 0 & 0 \\ -\beta\gamma[\beta] & \gamma[\beta] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

he boosted 4-vector is

```
Λx[β] . {t, x, y, z}
```

$$\left\{\frac{t}{\sqrt{1-\beta^2}} - \frac{x\beta}{\sqrt{1-\beta^2}}, \frac{x}{\sqrt{1-\beta^2}} - \frac{t\beta}{\sqrt{1-\beta^2}}, y, z\right\}$$

$$\Lambda_x[\beta]\,(t, x, y, z) = \left(\frac{t}{\sqrt{1-\beta^2}} - \frac{x\beta}{\sqrt{1-\beta^2}}, \frac{x}{\sqrt{1-\beta^2}} - \frac{t\beta}{\sqrt{1-\beta^2}}, y, z\right)$$

To demonstrate that the interval has been preserved, calculate the inner product.

```
Simplify[
  Λx[β] . {t, x, y, z}.Mmetric.Λx[β] . {t, x, y, z}]
```

$$t^2 - x^2 - y^2 - z^2$$

$$\Lambda_x[\beta]\,(t, x, y, z)\, g_\mu{}^\nu\, \Lambda_x[\beta]\,(t, x, y, z) = t^2 - x^2 - y^2 - z^2$$

Starting from a 4-vector, this is the only way to boost a reference frame along the x axis to another 4-vector and preserve the inner product. However, it is not clear why one must necessarily start from a 4-vector.

Using quaternions in special relativity

Events are treated as quaternions, a skew field or division algebra that is 4 dimensional. Any tool built to manipulate quaternions will also be a quaternion. In this way, although events play a different role from operators, they are made of identical mathematical fabric.

The square of a quaternion is

```
Simplify[
 (q[t, x, y, z] . q[t, x, y, z] +
  q[t, x, y, z] . q[t, x, y, z]) / 2 . {1, 0, 0, 0}]
```

$$\{t^2 - x^2 - y^2 - z^2, 2\,t\,x, 2\,t\,y, 2\,t\,z\}$$

$$\left(t, \vec{X}\right)^2 = \left(t^2 - \vec{X}.\vec{X}, 2\,t\,\vec{X}\right)$$

The first term of squaring a quaternion is the invariant interval squared. There is implicitly, a form of the Minkowski metric that is part of the rules of quaternion multiplication. The vector portion is frame-dependent. If a set of quaternions can be found that do not alter the interval, then that set would serve the same role as the Lorentz group, acting on quaternions, not on 4-vectors. If two 4-vectors x and x' are known to have the property that their intervals are identical, then the first term of squaring q[x] and q[x'] will be identical. Because quaternions are a division ring, there must exist a quaternion L such that L q[x] = q[x'] since L = q[x'] q[x]ˆ-1. The inverse of a quaternion is its transpose divided by the square of the norm (which is the first term of transpose of a quaternion times itself). Apply this approach to determine L for 4-vectors boosted along the x axis.

```
Simplify[
  q[γ[β] t - β γ[β] x, -β γ[β] t + γ[β] x, y, z] .
   Transpose[q[t, x, y, z]]
  / (Transpose[q[t, x, y, z]] . q[t, x, y, z])[[1, 1]] .
  {1, 0, 0, 0}]
```

$$\left\{\frac{t^2+x^2-2\,t\,x\,\beta+(y^2+z^2)\sqrt{1-\beta^2}}{(t^2+x^2+y^2+z^2)\sqrt{1-\beta^2}},\ \frac{(-t^2+x^2)\,\beta}{(t^2+x^2+y^2+z^2)\sqrt{1-\beta^2}},\right.$$
$$\frac{-t\left(y+z\,\beta-y\sqrt{1-\beta^2}\right)+x\left(z+y\,\beta-z\sqrt{1-\beta^2}\right)}{(t^2+x^2+y^2+z^2)\sqrt{1-\beta^2}},$$
$$\left.\frac{x\left(z\,\beta+y\left(-1+\sqrt{1-\beta^2}\right)\right)+t\left(y\,\beta+z\left(-1+\sqrt{1-\beta^2}\right)\right)}{(t^2+x^2+y^2+z^2)\sqrt{1-\beta^2}}\right\}$$

$$(\gamma[\beta]\,t-\beta\,\gamma[\beta]\,x,\ -\beta\,\gamma[\beta]\,t+\gamma[\beta]\,x,\ y,\ z)\,(t,\ x,\ y,\ z)^{-1}$$

$$= \Big(t^2+x^2-2\,t\,x\,\beta+(y^2+z^2)\sqrt{1-\beta^2},\ (-t^2+x^2)\,\beta,$$
$$-t\left(y+z\,\beta-y\sqrt{1-\beta^2}\right)+x\left(z+y\,\beta-z\sqrt{1-\beta^2}\right),$$
$$x\left(z\,\beta+y\left(-1+\sqrt{1-\beta^2}\right)\right)+t\left(y\,\beta+z\left(-1+\sqrt{1-\beta^2}\right)\right)\Big)$$
$$\Big/(t^2+x^2+y^2+z^2)\sqrt{1-\beta^2} \equiv L$$

Define the Lorentz boost quaternion L along x using this equations. L depends on the relative velocity and position, making it "local" in a sense. See if L q[x] = q[x'].

```
Simplify[Expand[(
L[t, x, y, z, β] . q[t, x, y, z]) . {1, 0, 0, 0}]]
```

$$\{(t-x\,\beta)\,\gamma[\beta],\ (x-t\,\beta)\,\gamma[\beta],\ y,\ z\}$$

$$L[t,\ x,\ y,\ z,\ \beta]\,(t,\ x,\ y,\ z)$$
$$= (\gamma\,t-\gamma\,\beta\,x,\ -\gamma\,\beta\,t+\gamma\,x,\ y,\ z)$$

This is a quaternion composed of the boosted 4-vector. At this point, it can be said that _any_ problem that can be solved using 4-vectors, the Minkowski metric and a Lorentz boost along the x axis can also be solved using the above quaternion for boosting the event quaternion. This is because both techniques transform the same set of 4 numbers to the same new set of 4 numbers using the same variable beta. To see this work in practice, please examine the problem sets.

Confirm the interval is unchanged.

```
Simplify[(
    L[t, x, y, z, β] . q[t, x, y, z] .
    L[t, x, y, z, β] . q[t, x, y, z]) . {1, 0, 0, 0}]
```

$$\left\{t^2-x^2-y^2-z^2,\ \frac{2\,(t^2\,\beta+x^2\,\beta-t\,x\,(1+\beta^2))}{-1+\beta^2},\ \frac{2\,y\,(t-x\,\beta)}{\sqrt{1-\beta^2}},\ \frac{2\,z\,(t-x\,\beta)}{\sqrt{1-\beta^2}}\right\}$$

$$(L[t, x, y, z, \beta]\,(t, x, y, z))^2$$

$$= \left(t^2 - x^2 - y^2 - z^2, \frac{2\,(t^2\beta + x^2\beta - t\,x\,(1+\beta^2))}{-1+\beta^2}, \frac{2\,y\,(t - x\,\beta)}{\sqrt{1-\beta^2}}, \frac{2\,z\,(t - x\,\beta)}{\sqrt{1-\beta^2}}\right)$$

The first term is conserved as expected. The vector portion of the square is frame dependent.

Using quaternions in practice

The boost quaternion L is too complex for simple calculations. *Mathematica* does the grunge work. A great many problems in special relativity do not involve angular momentum, which in effect sets y = z = 0. Further, it is often the case that t = 0, or x = 0, or for Doppler shift problems, x = t. In these cases, the boost quaternion L becomes a very simple.

If t = 0, then

```
L == L[0, x, 0, 0, β] . {1, 0, 0, 0}
```

$$L = \left(\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right)$$

$$L = \gamma(1, \beta, 0, 0)$$

$$q -> q' = Lq$$

$$(0, x, 0, 0) -> (t', x', 0, 0) = (-\gamma\,\beta\,x, \gamma\,x, 0, 0)$$

If x = 0, then

```
L == L[t, 0, 0, 0, β] . {1, 0, 0, 0}
```

$$L = \left(\frac{1}{\sqrt{1-\beta^2}}, -\frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right)$$

$$L = \gamma(1, -\beta, 0, 0)$$

$$q -> q' = Lq$$

$$(t, \vec{0}) -> (t', x', 0, 0) = (\gamma\,t, -\gamma\,\beta\,t, 0, 0)$$

If t = x, then

```
L == Simplify[L[t, t, 0, 0, β] . {1, 0, 0, 0}]
```

$$L == \left\{\frac{1-\beta}{\sqrt{1-\beta^2}}, 0, 0, 0\right\}$$

$$L = \gamma(1-\beta, 0, 0, 0)$$

$$q -> q' = Lq$$

$$(t, x, 0, 0) -> (t', x', 0, 0) = \gamma(1-\beta)\,(t, x, 0, 0)$$

Note: this is for blueshifts. Redshifts have a plus instead of the minus.

Over 50 problems in a sophomore-level relativistic mechanics class have been solved using quaternions. 90% required this very simple form for the boost quaternion.

Implications

Problems in special relativity can be solved either using 4-vectors, the Minkowski metric and the Lorentz group, or using quaternions. No experimental difference between the two methods has been presented. At this point the difference is in the mathematical foundations.

An immense amount of work has gone into the study of metrics, particular in the field of general relativity. A large effort has gone into group theory and its applications to particle physics. Yet attempts to unite these two areas of study have failed.

There is no division between events, metrics and operators when solving problems using quaternions. One must be judicious in choosing quaternions that will be relevant to a particular problem in physics and therein lies the skill. Yet this creates hope that by using quaternions, the long division between between metrics (the Grassman inner product) and groups of transformations (sets of quaternions that preserve the Grassman inner product) may be bridged.

3 8.033 Problem Set 1, Kinematic Effects of Relativity

Preamble: Initiation functions

There are a few tools required to solve problems in special relativity using quaternions to characterize events in spacetime. The most basic are gamma and a round value for c.

$$\gamma[\beta\_] := \frac{1}{\sqrt{1 - \beta^2}}$$

$$c = 3\ 10^8\ m/s;$$

Define a function for quaternions using its matrix representation.

$$q[t\_, x\_, y\_, z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

A quaternion L that perform a transform on a quaternion -

L q[**x**] = q[**x'**] - identical to how the Lorentz transformation acts on 4-vectors - Lambda **x** = **x'** - should exist. These are described in detail in the notebook "A different algebra for boosts." For boosts along the x axis with y = z = 0, the general function for L is

$$L[t\_, x\_, \beta\_] := \frac{1}{t^2 + x^2} q[\gamma[\beta]\, t^2 - 2\gamma[\beta]\beta\, t\, x + \gamma[\beta]\, x^2, -\beta\gamma[\beta]\,(t^2 - x^2), 0, 0]$$

Most of the problems here involve much simpler cases for L, where t or x is zero, or t is equal to x.

If t = 0, then

$$L[0, x, \beta] . \{1, 0, 0, 0\}$$

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If x = 0, then

$$L[t, 0, \beta] . \{1, 0, 0, 0\}$$

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, -\frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If t = x, then

$$\text{Simplify}[L[t, t, \beta] . \{1, 0, 0, 0\}]$$

$$\left\{\frac{1-\beta}{\sqrt{1-\beta^2}}, 0, 0, 0\right\}$$

Note: this is for blueshifts. Redshifts have a plus instead of the minus.

The problems are from "Basic Concepts in Relativity" by Resnick and Halliday, ©1992 by Macmillian Publishing, "Special Relativity" by A. P. French, © 1966, 1968 by MIT, and Prof. M. Baranger of MIT.

R&H 2-9: A moving clock

Q: A clock moves along the x axis at a speed of 0.6c and reads zero as it passes the origin. What time does it read as it passes the 180 m mark on the x axis?

A: A clock measures an interval between two events. The first event occurs at the origin. The second event happens at 180 m in a time of 180 m/v. Calculate the interval by squaring the difference quaternion and then taking the square root of the first term.

```
Sqrt[(q[180 m/(0.6 c), 180 m/c, 0, 0] .
 q[180 m/(0.6 c), 180 m/c, 0, 0])[[1, 1]]]
```

$8. \times 10^{-7} \sqrt{s^2}$

The moving clock reads 8 x 10ˆ − 7 seconds.

R&H 2-10: A moving rocket

Q: A rod lies parallel to the x axis of reference frame S, moving along this axis at a speed of 0.6c. Its rest length is 1.0 m. What will be its measured length in frame S?

A: Consider the meter stick at rest in a frame S', one end at the origin, the other at q[0, 1 m, 0, 0]. We want to boost the stick end quaternion to frame S. The boost quaternion when t=y=z=0 is ***L = q[γ, -γ β, 0, 0]***. In frame S', frame S is moving at -0.6c.

```
stick_end = q[γ[-0.6], -0.6 γ[-0.6], 0, 0] . q[0, 1 m, 0, 0];
```

The start of the stick will move for a time equal to the first term of the boosted quaternion, and moved by a distance `x = vt/c`.

```
stick_start = q[stick_end[[1, 1]], 0.6 stick_end[[1, 1]], 0, 0];
```

The meter stick's length in frame S will be the difference at the same instant in this frame between the boosted stick end and translocated stick start.

```
stick_length = (stick_end - stick_start).{1, 0, 0, 0}
```

{0. m, 0.8 m, 0, 0}

The meter stick is length contracted to 0.8 meters in frame S.

R&H 2-13: A fast spaceship

Q: The length of a spaceship is measured to be exactly half its rest length. (a) What is the speed of the spaceship relative to the observer's frame? (b) By what factor does the spaceship's clocks run slow, compared to clocks in the observer's frame?

A: (a) Consider the spaceship at rest, one end at the origin, the other at q[0, d, 0, 0]. We want to boost the ship end quaternion to the observer's frame. The boost quaternion when t=y=z=0 is ***L = q[γ, γ β, 0, 0]***. In the ship's frame, the observer is moving at `-v/c`.

```
ship_end = q[γ[-β], -β γ[-β], 0, 0] . q[0, d, 0, 0];
```

The start of the ship will move for a time equal to the first term of the boosted quaternion, and moved by a distance `x = vt/c`.

```
ship_start = q[ship_end[[1, 1]], β ship_end[[1, 1]], 0, 0];
```

The ship's length in the observer's frame will be the difference at the same instant in this frame between the boosted ship end and translocated ship start.

```
ship_length = (ship_end - ship_start).{1, 0, 0, 0}
```

$$\left\{0, \frac{d}{\sqrt{1-\beta^2}} - \frac{d\,\beta^2}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

Solve for `beta` setting this distance to d/2.

```
Solve[ship_length[[2]] == d/2, β]
```

$$\left\{\left\{\beta \to -\frac{\sqrt{3}}{2}\right\}, \left\{\beta \to \frac{\sqrt{3}}{2}\right\}\right\}$$

Beta is $\sqrt{3}/2 = 0.866$.

(b) The factor that the clocks appear to run at different rates is gamma.

$$\gamma\left[\frac{\sqrt{3}}{2}\right]$$

2

4 8.033 Problem Set 2, More Kinematic Effects of Relativity

French: 4-5

Q: A rocketship of proper length d travels at constant velocity v relative to a frame S. The nose of the ship (A') passes the point A in S at t = t' = 0, and at this instant a light signal is sent out from A' to B' (the end of the ship). (a) When, by rocketship time (t'), does the signal reach the tail (B') of the ship? (b) At what time t1, as measured in S, does the signal reach the tail (B') of the ship?

(c) At what time t2, as measured in S, does the tail of the ship (B') pass the point A?

A: (a) In the rocket's frame, the light is emitted a proper length d from the origin traveling at c, so t' = d/c.

(b) In the rocket's frame, the event of the signal reaching the tail is represented by the quaternion **q[d/C, d/C, 0, 0]**. In frame S, the light is blueshifted because the rocket is approaching at a speed of -beta.

```
t1 = Simplify[
    (γ[β] - β γ[β]) q[d/C, d/C, 0, 0] . {1, 0, 0, 0}]
```

$$\left\{-\frac{d\,(-1+\beta)}{C\sqrt{1-\beta^2}},\ -\frac{d\,(-1+\beta)}{C\sqrt{1-\beta^2}},\ 0,\ 0\right\}$$

The time the signal arrives in frame S is $t1 = \sqrt{\frac{1-\beta}{1+\beta}}\,\frac{d}{c}$.

(c) The length of the ship in frame S must be calculated first. Boost the ship's end at q[0, d, 0, 0] to frame S. The boost quaternion is **L = q[γ, γβ, 0, 0]**.

```
ship_end = q[γ[-β], -β γ[-β], 0, 0] . q[0, d, 0, 0];
```

The start of the ship will move for a time equal to the first term of the boosted quaternion, and moved by a distance x = vt/c.

```
ship_start = q[ship_end[[1, 1]], β ship_end[[1, 1]], 0, 0];
```

The ship's length in frame S will be the difference at the same instant in this frame between the boosted ship end and translocated ship start.

```
ship_length = (ship_end - ship_start).{1, 0, 0, 0}
```

$$\left\{0,\ \frac{d}{\sqrt{1-\beta^2}} - \frac{d\,\beta^2}{\sqrt{1-\beta^2}},\ 0,\ 0\right\}$$

```
t2 = Simplify[ship_length[[2]] / β]
```

$$\frac{d - d\,\beta^2}{\beta\sqrt{1-\beta^2}}$$

The time the rocketship's tail arrives is $t2 = d/\beta\gamma$ in frame S.

French: 4-9

Q: Two spaceships, each measuring 100 m in its own rest frame, pass by each other traveling in opposite directions. The instruments on spaceship A determine that the front end of spaceship B requires 5 microseconds to traverse the full length of A. (a) What is the relative velocity of the 2 spaceships? (b) A clock in the front end of B reads exactly one o'clock as it passes by the front end of A. What will the clock read as it passes by the rear end of A?

A: (a) Given a length and a time, divide one by the other to get the relative velocity.

$$\frac{q[5\,10^{-6}\,s,\ 100\,m,\ 0,\ 0][[2, 1]]}{q[5\,10^{-6}\,s,\ 100\,m,\ 0,\ 0][[1, 1]]}$$

$$\frac{20000000\,\text{m}}{\text{s}}$$

The relative velocity is 2 x 10ˆ7 m/s.

(b) The proper time of the clock in rocketship B is the interval, which when using quaternions is the square root of the first term of the quaternion squared.

```
Sqrt[(q[5. 10^-6 s, 100 m/c, 0, 0] .
  q[5. 10^-6 s, 100 m/c, 0, 0])[[1, 1]]]
```

$4.98888 \times 10^{-6} \sqrt{s^2}$

The clock in rocket B reads one o'clock plus 4.99 microseconds.

French: 4-12

Q: At noon a rocketship passes the Earth with a velocity 0.8c. Observers on the ship and on the Earth agree that it is noon.

(a) At 12:30 P.M. as read by a rocketship clock, the ship passes an interplanetary navigational station that is fixed relative to the Earth and whose clocks read Earth time. What time is it at the station? (b) How far from the Earth (in Earth coordinates) is the station?

(c) At 12:30 P.M. rocketship time the ship reports by radio back to Earth. When (by Earth time) does the Earth receive the signal?

(d) The station on Earth replies immediately When (by rocket time) is the reply received?

Solve this problem TWICE, once using the Earth as a reference frame and then using the rocket at the reference frame.

A: (a) From the Earth frame, we are given the proper time on the rocket clock as 30'. This interval is equal to the one seen by the Earth, which is calculated by squaring the quaternion and solving for t.

```
Solve[(q[t, -0.8 t, 0, 0] .
          q[t, -0.8 t, 0, 0])[[1, 1]] == (30 min)^2,
  t]
{{t → -50. min}, {t → 50. min}}
```

The time on the Earth clock is 50 min.

(b) Multiply the time by the speed and get the units right.

$$d_{station} = 0.8\,c\;50\,\text{min}\,\frac{60\,\text{s}}{\text{min}}$$

7.2×10^{11} m

The distance is 7.2 x 10ˆ11 meters.

(c) The time of the rocket emitting the signal, 50', plus its travel time from that location, 50' v/c = 40, is 90', or 1:30.

$$t_{earthreceives} = 50\,\text{min} + 0.8\;50\,\text{min}$$

90. min

(d) Find the intersection of the world line of the rocket, x/c = v t/c, and the world line of the light emitted from the Earth at 90 min,

x/c = t - 90'.

```
Solve[x/C == t - 90 min /. x/C-> 0.8 t, t]
{{t → 450. min}}
```

The position of the event is 450' v/c = 360'. We need the proper time of this interval, which will be the time on the rocket clock.

```
√((q[450 min, 360 min, 0, 0] .
  q[450 min, 360 min, 0, 0])[[1, 1]])
270 √min²
```

At 4:30 rocket time, the light from the Earth will be received at the rocket.

A': Now from the rocket frame...

(a') From the rocket frame, we are given t=30', x = 0. We need to boost this proper time interval to the Earth's frame.

```
q[γ[0.8], 0.8 γ[0.8], 0, 0] .
q[30 min, 0, 0, 0] . {1, 0, 0, 0}
{50. min, 40. min, 0, 0}
```

The time in the Earth frame is 50 min.

(b') The distance from the Earth in the Earth's frame is the second term of the above quaternion. Convert 40 min to meters.

$$40\,\mathrm{min}\,\frac{60\,\mathrm{s}}{\mathrm{min}}\,c$$

```
720000000000 m
```

The distance is 7.2 x 10ˆ11 meters.

(c') Find the intersection of the world line of the Earth, x/c = -v t/c, and the light emitted at 30', x/c = -t + 30.

```
Solve[x/C == -t + 30 min /. x/C -> -0.8 t, t]
{{t → 150. min}}
```

The position of this event is 150' v/c = 120'. We need the proper time of this interval, which will be the time on the Earth clock.

```
√((q[150 min, 120 min, 0, 0] .
  q[150 min, 120 min, 0, 0])[[1, 1]])
90 √min²
```

At 1:30 Earth time, the light will be received from the rocket.

(d') It is 150' in the rocket frame when the Earth emits the signal. It will take 120' for the signal to arrive. 150' + 120' = 270' or 4:30.

```
q[150 min, 120 min, 0, 0][[1, 1]] +
q[150 min, 120 min, 0, 0][[2, 1]]
270 min
```

The same answer again!

R & H: 2-14 A slow airplane

Q: An airplane whose rest length is 40.0 m is moving at a uniform velocity with respect to the Earth at a speed of 630 m/s. (a) By what fraction of its rest length will it appear to be shortened to an observer on Earth? (b) How long would it take by Earth clocks for the airplane's clock to fall behind by 1 microsecond, assuming that only special relativity applies?

A: (a) Boost the plane's tail in the plane's frame to the Earth's frame by a speed of -630 m/s.

```
βplane = 630. m/s/c
```

```
2.1 × 10^-6

d_tail = q[γ[-β_plane], -β_plane γ[-β_plane], 0, 0] .
    q[0, 40 m, 0, 0];
```

Calculate the distance traveled by the nose in this amount of time.

```
d_nose = q[d_tail[[1, 1]], β_plane d_tail[[1, 1]], 0, 0];
```

Subtract the distance traveled by the nose from the tail. Take the ratio of this difference with the rest length.

$$1 - \frac{(d_{tail} - d_{nose})[[2,1]]}{40\,m}$$

2.205×10^{-12}

The ratio of lengths as seen on the Earth is 1 minus this small number.

(b) We want to know the differential time between a boosted clock and one at rest. This is the first term of the difference between a boosted and unboosted clock.

```
(q[γ[β_plane], β_plane γ[β_plane], 0, 0] . q[t, 0, 0, 0] -
    q[t, 0, 0, 0])[[1, 1]]
```

$2.205 \times 10^{-12}\, t$

Set this equal to 1 microsecond and solve for t.

```
NSolve[% == 10^-6 s, t]
{{t → 453515. s}}
```

$$453515.\,s \frac{min}{60\,s} \frac{hr}{60\,min} \frac{day}{24\,hr}$$

5.24902 day

The plane must travel for 4.53x10ˆ5 s to get out of sync by a microsecond with the Earth, or 5.25 days.

R & H: 2-21 Travel to the galactic center!

Q: (a) Can a person, in principle, travel from Earth to the galactic center (which is about 28,000 lyr distant) in a normal lifetime? (b) What constant velocity would be needed to make the trip in 30 years (proper time)?

A: (b) Boost the rocketeer up to the BIG speed B=1-e, set the distance to the destination d, and solve for e.

```
Solve[
  (q[γ[1 - e], (1 - e) γ[1 - e], 0, 0] .
  q[t, 0, 0, 0])[[2, 1]] == d, e]
```

$$\left\{\left\{e \to \frac{d^2 + t^2 - d\sqrt{d^2+t^2}}{d^2+t^2}\right\}, \left\{e \to \frac{d^2 + t^2 + d\sqrt{d^2+t^2}}{d^2+t^2}\right\}\right\}$$

Plug in numbers.

```
N[%/. {d-> 28000, t-> 30}]
```

$\{\{e \to 5.73979 \times 10^{-7}\}, \{e \to 2.\}\}$

The constant speed required to make the trip in 30 years is 1 - 5.7 x 10ˆ-7 less than c. The answer to (a) is that as a purely mathematical exercise, one could say yes. However, this does not account for the energy required to reach such a speed. An analysis which investigated the energy requirements would probably conclude that it cannot be done.

R & H: 2-24 Decay in flight (II)

Q: The mean lifetime of muons stopped in a lead block in the laboratory is measured to be 2.2 microseconds. The mean lifetime of high-speed muons in a burst of cosmic rays observed from the Earth is measured to be 16 microseconds. Find the speed of these cosmic ray muons.

A: Boost the muon from its rest frame to the lab.

```
muon_lab = q[γ[β], β γ[β], 0, 0] . q[2.2 μs, 0, 0, 0];
```

Set the time component of the quaternion equal to 16 microseconds.

```
Solve[muon_lab[[1, 1]] == 16 μs, β]
{{β → -0.990502}, {β → 0.990502}}
```

The muon is travelling 0.9905 c.

R & H: 2-25 Decay in flight (III)

Q: An unstable high-energy particle enters a detector and leaves a track 1.05 mm long before it decays. Its speed relative to the detector was 0.992c. What is its proper lifetime?

A: Boost the proper path of unknown length L by v/c = 0.992, solve for L given the lab length L'.

```
Solve[(q[γ[-0.992], -(-0.992) γ[-0.992], 0, 0] .
        q[0, L, 0, 0])[[2, 1]] == 0.00105 m/c, L]
{{L → 4.41833 × 10^-13 s}}
```

The average lifetime is 4.4 x 10ˆ − 13 s.

R & H: 2-26 Decay in flight (IV)

Q: In the target area of an accelerator laboratory there is a straight evacuated tube 300 m long. A momentary burst of 1 million radioactive particles enters at one end of the tube, moving at a speed of 0.80c. Half of them arrive at the other end without having decayed. (a) How long is the tube as measured by an observer moving with the particles? (b) What is the half-life of the particles in this same reference frame? (c) With what speed is the tube measured to move in this frame?

A: (a) Same as above.

```
Solve[(q[γ[-0.8], -(-0.8) γ[-0.8], 0, 0] .
           q[0, L, 0, 0])[[2, 1]] == 300 m, L]
{{L → 180. m}}
```

The tube looks 180 m long to the moving particles.

(b) The length of the target is equal to one half life, t = L/v.

$$\frac{180\,\mathrm{m}}{0.8\,c}$$

7.5×10^{-7} s

The half life is 750 nanoseconds.

(c) By symmetry, v = 0.8c. By calculation.

$$\frac{180\,\mathrm{m}}{c\;7.5\;10^{-7}\,\mathrm{s}}$$

0.8

The tube looks like it is moving 0.8c in the rest frame of the particles.

R & H: 2-28 Simultaneous - but to whom?

Q: An experimenter arranges to trigger two flashbulbs simultaneously, a blue flash located at the origin of his reference frame and a red flash at x = 30 km. A second observer is moving at a speed of 0.25c in the direction of increasing x, and also views these flashes. (a) What time interval between them does he find?

(b) Which flash does he say occurs first?

A: (a) For the first observer, the blue flash stays at the origin. The red flash is boosted to a new location in spacetime.

```
(q[γ[0.25], (0.25) γ[0.25], 0, 0] .
  q[0, 30 10^3 m/c, 0, 0])[[1, 1]]
-0.0000258199 s
```

There will be 26 microseconds between the flashes.

(b) The origin won't change under the boost. From part (a) the flash of red light event will be changed to -26 microseconds. Therefore the red light appears first to the rocketeer.

R & H: 2-36 What time is it anyway?

Q: Observers S and S' stand at the origins of their respective frames, which are moving relative to each other with a speed of 0.6c. Each has a standard clock, which, as usual, they set to zero when the two origins coincide. Observer S keeps the S' clock visually in sight. (a) What time will the S' clock record when the S clock records 5 microseconds? (b) What time will observer S actually read on the S' clock when his own clock reads 5 microseconds?

A: (a) We must determine the proper time for a clock with t = 5 microseconds, and x = v t, by taking the square root of the first term of the event quaternion squared.

$$\sqrt{(q[5\,\mu s, 0.6\ 5\,\mu s, 0, 0] . q[5\,\mu s, 0.6\ 5\,\mu s, 0, 0])[[1, 1]]}$$

$4.\ \sqrt{\mu s^2}$

The S' clock will record 4 microseconds when the clock in S reaches 5 microseconds.

(b) The intersection of the worldline of the rocket, x/c = 0.6 t and a lightcone passing through t = 5 microseconds, x = 0 can be solved for t.

```
Solve[x/C == -t + 5 μs /. x/C-> 0.6 t, t]
{{t → 3.125 μs}}
```

The S' clock will read the interval of the quaternion at this intersection. Calculate the interval as in part (a).

$$\sqrt{(q[3.125\,\mu s, 0.6\ 3.125\,\mu s, 0, 0] . q[3.125\,\mu s, 0.6\ 3.125\,\mu s, 0, 0])[[1, 1]]}$$

$2.5\ \sqrt{\mu s^2}$

At 5 microseconds, the observer in frame S will actually see 2.5 microseconds on the S' clock.

Baranger: The cat's life

Q: A newborn cat is put aboard a ship leaving Earth for Andromeda at speed v = 0.6c. The cat dies on the ship at age 7 years. (a) How far from the Earth in the Earth's frame is the ship when the cat dies? (b) A radio signal is sent from the ship when the cat dies. When does this signal get to the Earth by Earth time?

(c) Bonus: What is the probability amplitude that Schrodinger killed the cat?

A: (a) The proper time of the cat's life is 7 years. Boost it to the Earth's frame.

```
q[γ[-0.6], -(-0.6) γ[-0.6], 0, 0] .
q[7 yr, 0, 0, 0] . {1, 0, 0, 0}
{8.75 yr, 5.25 yr, 0, 0}
```

In the Earth's frame, the cat died after traveling a distance equal to 5.25 years.

(b) It will take 5.25 years for the light to get back from the time when the cat died (8.75 years), so the signal reaches Earth in

5.25 + 8.75 = 14 years.

(c) Schrodinger posed the question as a joke. He is *definitely* still laughing.

Baranger: A particle's life

Q: A particle moving with speed v = 0.99c goes on the average a distance 12.5 m before decaying. What is its proper lifetime?

A: Take the lifetime of the particle in its own frame, boost it to the lab's frame.

```
particle_lab =
     q[γ[0.99], 0.99 γ[0.99], 0, 0] . q[t, 0, 0, 0];
```

In the lab, x = 12 m. Set them equal, solve for the lifetime.

```
Solve[particle_lab[[2, 1]] == 12 m/c, t]
{{t → 5.69969 × 10^-9 s}}
```

The lifetime is 5.7 ns.

Baranger: Trains & clocks

Q: The train is moving with a velocity v. At the head of the train, the engineer compares her clock C'1 with a stationary clock C1 outside as she passes it, and finds that both clocks read time zero. At the same moment (for the train frame) the conductor in the caboose compares his clock C'2 (which therefore also reads zero) with a stationary clock C2 he happens to be passing. What does C2 read? The distance between the clocks C'1 and C'2 measured by people on the train is L.

A: The interval for both sets of clocks is L/C. For the observer on the ground, set the time to t, the distance to vt/c. Square this quaternion, set the first term equal to the square of the interval, and solve for t.

```
Solve[Sqrt[(q[t, v t/C, 0, 0] .
     q[t, v t/C, 0, 0])[[1, 1]]] == L/C, t]
```

$$\left\{\left\{t \to -\frac{I\,L}{\sqrt{-C^2+v^2}}\right\}, \left\{t \to \frac{I\,L}{\sqrt{-C^2+v^2}}\right\}\right\}$$

The clock will read $t = -\gamma L/c$. Note that *Mathematica* has erroneously injected a factor of I into the "solution".

Baranger: Blow up the Earth

Q: Some inhabitants of the Andromeda nebula are traveling through the Milky Way in a flying saucer whose constant velocity equals 0.8c. Going by the Earth, they find out that it is A. D. 1996 here and they synchronize their clocks with ours. In A. D. 2005, mankind blows up the Earth. At what time, on their clock, do the travellers in the flying saucer learn of this event, assuming that they have been watching us all along through a telescope. Try a few ways of doing this problem.

A: From the frame of the Earth, find the intersection of the world line of the saucer, x/c = 0.8 t, and the light cone from the explosion of the Earth, x/c = t + 9 yr.

```
Solve[x/C == t - 9 yr /. x/C -> 0.8 t, t]
{{t → 45. yr}}
```

The saucer has travelled a distance d = v t. Calculate the interval which will give the saucer's proper time.

$$\sqrt{(q[45\,yr,\ 0.8\,45\,yr,\ 0,\ 0]\ .\ q[45\,yr,\ 0.8\,45\,yr,\ 0,\ 0])[[1,1]]}$$

$$27.\sqrt{yr^2}$$

In 27 years time, or 2023, the saucer will note the demise of Earth.

A': Repeat the calculation from the saucer frame. We know the interval is 9 years.

```
Solve[(q[t, -0.8t, 0, 0] .
            q[t, -0.8t, 0, 0])[[1, 1]] == (9 yr)^2, t]
{{t → -15. yr}, {t → 15. yr}}
```

The position will be x = v t = 12 years, which will take another twelve years to return, for a total of 27 years.

Post Ramble: Initialization functions

There are a few tools required to solve problems in special relativity using quaternions to characterize events in spacetime. The most basic are a round value for c and gamma.

$$c = 3\ 10^8\ m/s;$$

$$\gamma[\beta\_] := \frac{1}{\sqrt{1-\beta^2}}$$

Define a function for quaternions using its matrix representation.

$$q[t\_,\ x\_,\ y\_,\ z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

A quaternion L that transforms a quaternion (L q[**x**] = q[**x'**]) identical to how the Lorentz transformation acts on 4-vectors

(Lambda **x** = **x'**) should exist. These are described in detail in the notebook "A different algebra for boosts." For boosts along the x axis with y = z = 0, the general function for L is

$$L[t\_,\ x\_,\ \beta\_] := \frac{1}{t^2+x^2}\,q[\gamma[\beta]\,t^2 - 2\,\gamma[\beta]\,\beta\,t\,x + \gamma[\beta]\,x^2,\ -\beta\,\gamma[\beta]\,(t^2-x^2),\ 0,\ 0]$$

Most of the problems here involve much simpler cases for L, where t or x is zero, or t is equal to x.

If t = 0, then

$$L[0,\ x,\ \beta]\ .\ \{1,\ 0,\ 0,\ 0\}$$

$$\left\{\frac{1}{\sqrt{1-\beta^2}},\ \frac{\beta}{\sqrt{1-\beta^2}},\ 0,\ 0\right\}$$

If x = 0, then

```
L[t, 0, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, -\frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If t = x, then

```
Simplify[L[t, t, β] . {1, 0, 0, 0}]
```

$$\left\{\frac{1-\beta}{\sqrt{1-\beta^2}}, 0, 0, 0\right\}$$

Note: this is for redshifts. Blueshifts have a plus instead of the minus.

5 8.033 Problem Set 3: The Lorentz transformation and addition of velocities

Lorentz transformations

Baranger: Inverse of a boost

Q: Start from the 4 Lorentz equations giving the variables T, X, Y, and Z in terms of t, x, y and z, and solve these equations for t, x, y and z in terms of T, X, Y, and Z. Make sure the result is what you expected.

A: The problem will be solved for a boost transformation without any angular momentum (the case for y and z not equal to zero just gets messier, not deeper).

```
Simplify[L[t, x, β] . q[t, x, 0, 0] . {1, 0, 0, 0}]
```

$$\left\{\frac{t - x\beta}{\sqrt{1-\beta^2}}, \frac{x - t\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

L q = q', so q = L^-1 q'. L's inverse always exists and is straight forward to calculate: the inverse is the transpose divided by the square of the norm.

```
Simplify[Transpose[L[T, X, β]]/
(L[T, X, β][[1, 1]]^2 + L[T, X, β][[2, 1]]^2)];

Simplify[% . q[γ[β] T - βγ[β] X, -βγ[β] T + γ[β] X, 0, 0] .
                        {1, 0, 0, 0}]
{T, X, 0, 0}
```

(The inverse was big and ugly, that's why it was hidden from view, but it does work! The things that change are the sign of the second component and the normalization factor which is quite bulky).

Baranger: Boosting photons

Q: In frame S, a flash of light is emitted at the origin and is absorbed on the x axis at x = d. Answer the following questions from the point of view of frame S', moving in the standard way: (a) What is the spatial separation d' between the point of emission and the point of absorption of the light? (b) How much time elapses between the emission and the absorption?

A: (a) In frame S, the point of absorption is q[d, d, 0, 0]. Boost this to frame S', and look at the x component.

```
(γ[β] + βγ[β]) q[d, d, 0, 0] . {1, 0, 0, 0}
```

$$\left\{d\left(\frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}\right), d\left(\frac{1}{\sqrt{1-\beta^2}} + \frac{\beta}{\sqrt{1-\beta^2}}\right), 0, 0\right\}$$

The separation is $d' = \gamma(1+\beta)d$, presuming frame S' is moving in the same direction as the light.

(b) The elapsed time is the first component of the transformed quaternion from above, or $t' = \gamma(1+\beta)d/c$.

Baranger: The ladder in the barn

Q: (long) Manuel wants to demonstrate the Lorentz length contraction by fitting his 10 m ladder inside his 8 m garage. He ties the ladder on top of his station wagon and asks his friend Linda to drive the wagon into the open door of the garage. Linda drives it at the speed of 0.8c, than therefore gamma = 5/3, which makes the ladder only 6 m long. Manuel is standing by the door and, as soon as the rear end of the ladder has passed, he shuts the door. He now has the entire ladder inside the 8 m garage, just as he said he would. However, this garage also has a back door and, since the brakes on the wagon are a little worn out, Manuel has instructed his other friend Gwen to open the back door at the precise instant that the front end of the ladder is about to hit it. It's OK, says Manuel, because the ladder was actually inside the garage, with both doors shut, for a finite amount of time, and that is all he wanted to prove. But it turns out that Gwen's cat, CloudNine, was sitting all the time on the ladder on top of the wagon, and CloudNine disagrees totally. He says: the ladder was really 10 m, while the garage was Lorentz contracted and only 8 m x 3/5 = 4.8 m, and obviously that the ladder was never, never totally inside the garage. CloudNine looks in good health; the ladder is still in one piece. What's going on? Who is right?

INSTRUCTIONS: Define 4 separate events. Choose coordinates in two frames. Elaborate the 2 descriptions of what happened, Manuel's and CloudNine's, giving precise numbers for all the events. Show that these two descriptions are actually totally consistent, given the known laws of special relativity.

A. From the viewpoint of the barn, the ladder is contracted. This involves boosting the end of the ladder, and subtracting where the start of the ladder has moved to at a simultaneous time.

```
Ladder_end = q[γ[-0.8], -0.8 γ[-0.8], 0, 0] . q[0, 10 m, 0, 0];

Ladder_start = q[Ladder_end[[1, 1]], 0.8 Ladder_end[[1, 1]], 0, 0];

Ladder_length = (L_end - L_start) . {1, 0, 0, 0}
{0. m, 6. m, 0, 0}
```

Manuel observes (correctly) that the ladder appears to be length contracted to 6 m in his reference frame.

Repeat this exercise for the cat frame looking at the length of the barn.

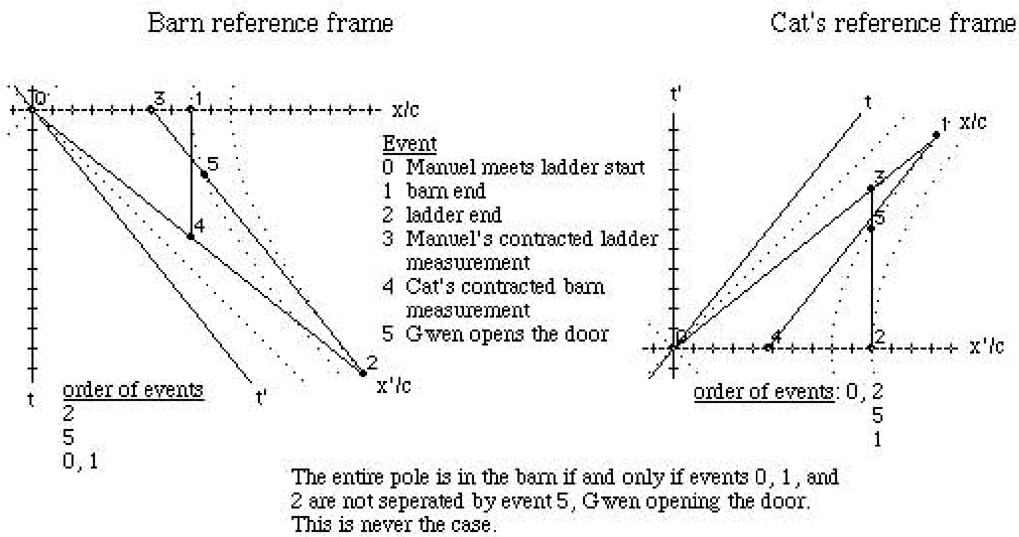
```
Barn_end = q[γ[0.8], 0.8 γ[0.8], 0, 0] . q[0, 8 m, 0, 0];

Barn_start = q[Barn_end[[1, 1]], -0.8 Barn_end[[1, 1]], 0, 0];

Barn_length = (Barn_end - Barn_start) . {1, 0, 0, 0}
{0. m, 4.8 m, 0, 0}
```

The barn is only 4.8m long from the cat's reference frame.

Manuel and the cat make correct statements about length contraction from their own reference frames.

What makes this problem confusing to discuss is that the doors of the barn and the ends of the ladder are spacelike separated, so the order of events in time can be reversed depending on the reference frame.

The reference event chosen for the following spacetime diagrams (created in the program Spacetime by Prof. Edwin F. Taylor) is Manuel meeting the start of the ladder. The end of the barn and the end of the ladder must be located on invariant hyperbolae 8 and 10 meters long respectively. Gwen must open the door somewhere on the barn's 8 meter hyperbola. Manuel's dream will only come true if the reference event and the two ends are not separated by Gwen's action. This is not the case as seen in the figure below:



Gwen always opens the far barn door before both ends of the ladder and barn are together with Manuel and the start of the ladder, averting disaster. The barn stays in one piece and we have had fun with moving objects with spacelike separations.

R&H: 2-34 2 flashes at different places - or are they?

Q: An observer S sees a flash of red light 1200 m from his position and a flash of blue light 720 m closer to him and on the same straight line. He measures the time interval between the occurrences of the flashes to be 5 microseconds, the red flash occurring first. (a) What is the relative velocity **v** of a second observer S' who would record these flashes as occurring at the same place? (b) From the point of view of S', which flash occurs first? (c) What time between them would S' measure?

A: (a) Boost both red and blue lights by beta into frame S'.

```
blue = L[5. 10^-6 s, 480 m/c, β] . q[5. 10^-6 s, 480 m/c, 0, 0];

red = q[γ[β], β γ[β], 0, 0] . q[0, 1200 m/c, 0, 0];
```

Set the distances equal to each other (the second components of the boosted quaternions) and solve.

```
Solve[blue[[2, 1]] - red[[2, 1]] == 0, β]
{{β → -0.48}}
```

The second observer moves toward the first observer at a relative speed of 0.48c.

(b) Substitute the value for v/c into the boosted time for the events.

```
blue[[1, 1]] /. β -> -0.48
6.57495 × 10^-6 s

red[[1, 1]] /. β -> -0.48
2.18861 × 10^-6 s
```

The red light happens first for observer S'.

(c) Calculate the difference between the first two components of the quaternions.

```
blue[[1, 1]] - red[[1, 1]] /. β -> -0.48
4.38634 × 10^-6 s
```

The time difference is 4.39 microseconds.

Additions of velocities

If a quaternion is normalized to its interval, it becomes:

$$\frac{q}{t^2 - x^2 - y^2 - z^2} = q[\gamma, \gamma\beta_x, \gamma\beta_y, \gamma\beta_z]$$

If a quaternion is normalized to its interval and the first term which is gamma, the result is a quaternion characterizing the velocities:

$$\frac{q}{\gamma(t^2 - x^2 - y^2 - z^2)} = q[1, \beta_x, \beta_y, \beta_z]$$

This quaternion can be formed from any quaternion and boosted accordingly.

For this series of problems, we need a more general boost quaternion, one where y and z are not zero.

```
Simplify[q[γ[β] t - β γ[β] x, -β γ[β] t + γ[β] x, y, z] .
    Transpose[q[t, x, y, z]] . {1, 0, 0, 0} /
   (t^2 + x^2 + y^2 + z^2)]
```

$$\left\{\frac{t^2 + x^2 - 2\,t\,x\,\beta + (y^2 + z^2)\sqrt{1-\beta^2}}{(t^2 + x^2 + y^2 + z^2)\sqrt{1-\beta^2}},\right.$$
$$\frac{(-t^2 + x^2)\,\beta}{(t^2 + x^2 + y^2 + z^2)\sqrt{1-\beta^2}},$$
$$\frac{-t\left(y + z\,\beta - y\sqrt{1-\beta^2}\right) + x\left(z + y\,\beta - z\sqrt{1-\beta^2}\right)}{(t^2 + x^2 + y^2 + z^2)\sqrt{1-\beta^2}},$$
$$\left.\frac{x\left(z\,\beta + y\left(-1 + \sqrt{1-\beta^2}\right)\right) + t\left(y\,\beta + z\left(-1 + \sqrt{1-\beta^2}\right)\right)}{(t^2 + x^2 + y^2 + z^2)\sqrt{1-\beta^2}}\right\}$$

Define the function Lambda to do the general boost along x.

```
Λ[t_, x_, y_, z_, β_] :=
1/(t^2 + x^2 + y^2 + z^2) *
q[γ[β] t^2 - 2 β γ[β] t x + γ[β] x^2 + y^2 + z^2,
- β γ[β] (t^2 - x^2),
t y - x z + γ[β] (-t + β x) y + γ[β] (-β t + x) z,
x y + t z + γ[β] (β t - x) y + γ[β] (-t + β x) z]
```

R&H: 2-59 Watching the decay of a moving nucleus

Q: A radioactive nucleus moves with a uniform velocity of 0.050c along the x axis of a reference frame S fixed with respect to the laboratory. It decays by emitting an electron whose speed, measured in a reference frame S' moving with the nucleus, is 0.80c. Consider first the case in which the emitted electron travels (a) along the common x-x' axis and (b) along the y' axis and find, for each case, its velocity as measured in frame S. (c) Suppose, however, that the emitted electron, viewed now from frame S, travels along the y axis of that frame with a speed of 0.80c. What is its velocity as measured in frame S'?

A: (a) Boost the velocity quaternion by -0.05c, and keep it as a velocity quaternion by normalizing it with the resulting gamma.

```
Λ[1, 0.8, 0, 0, -0.05] . q[1, 0.8, 0, 0] . {1, 0, 0, 0} /
(Λ[1, 0.8, 0, 0, -0.05] . q[1, 0.8, 0, 0])[[1, 1]]
{1., 0.817308, 0., 0.}
```

The relative velocity in frame S is 0.817c along the x axis.

(b) Do that again, with y = 0.8.

```
Λ[1, 0, 0.8, 0, -0.05] . q[1, 0, 0.8, 0] . {1, 0, 0, 0} /
(Λ[1, 0, 0.8, 0, -0.05] . q[1, 0, 0.8, 0])[[1, 1]]
{1., 0.05, 0.798999, 0.}
```

The magnitude and angle of the velocity vector can be calculated.

$$\sqrt{0.05^2 + 0.799^2}$$

0.800563

$$\frac{0.05}{0.8005} \frac{180}{\pi}$$

3.57875

The velocity vector is 0.8005c 3.58 to the right of the y axis.

(c) Repeat the calculation, switching the sign of the boost.

```
Λ[1, 0, 0.8, 0, 0.05] . q[1, 0, 0.8, 0] . {1, 0, 0, 0} /
(Λ[1, 0, 0.8, 0, 0.05] . q[1, 0, 0.8, 0])[[1, 1]]
{1., -0.05, 0.798999, 0.}
```

A similar quaternion to b, so the velocity vector is 0.80056c, but 3.57 to the left of the y axis.

Baranger: Boosting boosted frames.

Q: (a) Frame S' moves with respect to frame S with velocity beta1 in the +x direction. Frame S" moves with respect to frame S' with velocity beta2 also in the + x direction. Frame S" moves with respect to frame S with velocity Beta, also in the +x direction. Let gamma1, gamma2, and Gamma be the 3 Lorentz factors corresponding to these 3 velocities, respectively. Prove the formula

$$\Gamma = (1 + \beta_1 \beta_2) \gamma_1 \gamma_2 .$$

(b) 2 identical particles are having a head-on collision. In their center-of-mass frame, each has a Lorentz factor gamma. Assume gamma >> 1. Now look at them in the Lab frame in which one of them is at rest, and call Gamma the Lorentz factor of the projectile particle. Show that Gamma is approximately equal to $2\gamma^2$.

A: (a) From the reference frame of S', beta1 is towards it (so is negative) and beta2 is away from S' (so it is positive).

$$q[\gamma[-\beta_1], -\beta_1 \gamma[-\beta_1], 0, 0] . q[1, 0, 0, 0] . \{1, 0, 0, 0\}$$

$$\left\{\frac{1}{\sqrt{1-\beta_1^2}}, -\frac{\beta_1}{\sqrt{1-\beta_1^2}}, 0, 0\right\}$$

$$\Gamma = \text{Simplify}[\; L[\gamma[-\beta_1], -\beta_1 \gamma[-\beta_1], \beta_2] . q[\gamma[-\beta_1], -\beta_1 \gamma[-\beta_1], 0, 0] . \{1, 0, 0, 0\}]$$

$$\left\{\frac{1+\beta_1 \beta_2}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}, \frac{-\beta_1-\beta_2}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}, 0, 0\right\}$$

The first term is the gamma being sought. Note that the second term divided by the first term gives the expected addition of relative velocities for frame S'. The sign is opposite for frame S.

(b) If gamma >> 1, use the approximation for beta of one minus epsilon, where epsilon is a small number. Plug into the results from part (a).

```
Γbig = Simplify[Γ/. {β1 -> 1 - ϵ, β2 -> 1 - ϵ}]
```

$$\left\{-\frac{2-2\epsilon+\epsilon^2}{(-2+\epsilon)\,\epsilon}, -\frac{2\,(-1+\epsilon)}{(-2+\epsilon)\,\epsilon}, 0, 0\right\}$$

Square gamma.

```
Simplify[γ[1 - ϵ]^2]
```

$$\frac{1}{2\epsilon-\epsilon^2}$$

Substitute back into Gamma big.

```
Γbig /. -1/((-2 + ϵ)ϵ) -> γ^2
```

$$\left\{\gamma^2\,(2-2\epsilon+\epsilon^2), -\frac{2\,(-1+\epsilon)}{(-2+\epsilon)\,\epsilon}, 0, 0\right\}$$

Gamma for the projectile particle is the first term of the above quaternion, approximately $2\,\gamma^2$.

French: 5-7 2 ways to double a boost

Q: An inertial system S1 has a constant velocity v1 along the x axis relative to an inertial system S. Inertial system S2 has a velocity v2 relative to S1. Two successive Lorentz transformations enable up to go from (t, x, y, z) to (t1, x1, y1, z1) and then from (t1, x1, y1, z1) to (t2, x2, y2, z2). Show that this gives the same result as a single Lorentz transformation from (t, x, y, z) to (t2, x2, y2, z2) provided we take the velocity of S1 relative to S as

$$v = \frac{v1 + v2}{1 + v1\,v2/c2}.$$

A: Boost once, then again.

```
Simplify[L[t, x, β1] . q[t, x, 0, 0] . {1, 0, 0, 0}]
```

$$\left\{\frac{t - x\beta_1}{\sqrt{1-\beta_1^2}}, \frac{x - t\beta_1}{\sqrt{1-\beta_1^2}}, 0, 0\right\}$$

```
Simplify[
L[γ[β1] (t - β1 x), γ[β1] (-t β1 + x), β2] .
q[γ[β1] (t - β1 x), γ[β1] (-t β1 + x), 0, 0] . {1, 0, 0, 0}]
```

$$\left\{\frac{t - x\beta_2 + \beta_1\,(-x + t\beta_2)}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}, \frac{x - t\beta_2 + \beta_1\,(-t + x\beta_2)}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}, 0, 0\right\}$$

Now boost once using the addition of velocities rule.

```
Simplify[PowerExpand[
    L[t, x, (β1 + β2)/(1 + β1 β2)] .
    q[t, x, 0, 0] . {1, 0, 0, 0}]]
```

$$\left\{\frac{t - x\beta_2 + \beta_1\,(-x + t\beta_2)}{(1+\beta_1\beta_2)\sqrt{1-\frac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}}}, \frac{x - t\beta_2 + \beta_1\,(-t + x\beta_2)}{(1+\beta_1\beta_2)\sqrt{1-\frac{(\beta_1+\beta_2)^2}{(1+\beta_1\beta_2)^2}}}, 0, 0\right\}$$

These two quaternions are identical, as expected.

Post Ramble

There are a few tools required to solve problems in special relativity using quaternions to characterize events in spacetime. The most basic are a round value for c and gamma.

```
c = 3 10^8 m/s;
γ[β_] := 1/Sqrt[1 - β^2]
```

Define a function for quaternions using its matrix representation.

$$q[t\_, x\_, y\_, z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

A quaternion L that transforms a quaternion (L q[**x**] = q[**x'**]) identical to how the Lorentz transformation acts on 4-vectors

(Lambda **x** = **x'**) should exist. These are described in detail in the notebook "A different algebra for boosts." For boosts along the x axis with y = z = 0, the general function for L is

$$L[t\_, x\_, \beta\_] := \frac{1}{t^2 + x^2} q[\gamma[\beta]\, t^2 - 2\,\gamma[\beta]\,\beta\, t\, x + \gamma[\beta]\, x^2, -\beta\,\gamma[\beta]\, (t^2 - x^2), 0, 0]$$

Most of the problems here involve much simpler cases for L, where t or x is zero, or t is equal to x.

If t = 0, then

```
L[0, x, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If x = 0, then

```
L[t, 0, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, -\frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If t = x, then

```
Simplify[L[t, t, β] . {1, 0, 0, 0}]
```

$$\left\{\frac{1-\beta}{\sqrt{1-\beta^2}}, 0, 0, 0\right\}$$

Note: this is for blueshifts. Redshifts have a plus instead of the minus.

The problems are from "Basic Concepts in Relativity" by Resnick and Halliday, ©1992 by Macmillian Publishing, "Special Relativity" by A. P. French, © 1966, 1968 by MIT, and Prof. M. Baranger of MIT.

6 8.033 Problem Set 4: The Doppler Effect, 4-Vector Invariants & the Twin Paradox

The Doppler Effect

French: 5-9

Q: Three identical radio transmitters A, B, and C each transmitting at the frequency ν_o in its own rest frame are moving as shown.

```
<---A B C--->
```

(a) What is the frequency of B's signals as received by C?

(b) What is the frequency of A's signals as received by C? There are at least three ways to solve this question. See if you can find two.

A: (a) Let the inverse of the frequency be the time t_o. Redshift it!

```
Simplify[(γ[β] + β γ[β]) q[t_o, t_o, 0, 0] . {1, 0, 0, 0}]
```

$$\left\{\frac{(1+\beta)\, t_o}{\sqrt{1-\beta^2}}, \frac{(1+\beta)\, t_o}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

The frequency is the inverse of the time component, or

$$\nu = \nu_o \sqrt{\frac{(1-\beta)}{1+\beta"}}$$

(b) We need another redshift of exactly the same size.

```
Simplify[(γ[β] + β γ[β])^2 q[to, to, 0, 0] . {1, 0, 0, 0}]
```

$$\left\{-\frac{to\,(1+\beta)}{-1+\beta}, -\frac{to\,(1+\beta)}{-1+\beta}, 0, 0\right\}$$

The frequency it the inverse of the time component, so $\nu = \nu_o \frac{(1-\beta)}{1+\beta}$.

(b') Another approach is to boost the initial event with a speed equal to the two boosts, which by the addition of velocity formula is shown below.

$$\beta\beta = \frac{2\beta}{1+\beta^2};$$

Redshift with this velocity and try to simplify.

```
PowerExpand[Cancel[
  Simplify[
    Expand[
      (γ[ββ] + ββ γ[ββ]) q[to, to, 0, 0] . {1, 0, 0, 0}]]]]
```

$$\left\{\frac{to\,(-1+\beta^2)}{(-1+\beta)^2}, \frac{to\,(-1+\beta^2)}{(-1+\beta)^2}, 0, 0\right\}$$

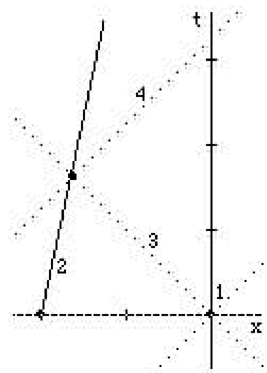
A step away from the previous result.

French: 5-10

Q: A pulsed radar source is at rest at the point x = 0. A large meteorite moves with constant velocity v toward the source; it is at the point x = -d at t = 0. A first radar pulse is emitted by the source at t = 0, and a second pulse at t = to (to << d/c). The pulses are reflected by the meteorite and return to the source. (a) Draw in spacetime graph (1) the source, (2) the meteorite, (3) the two outgoing pulses, (4) the reflected pulses. (b) Evaluate the time interval between the arrivals at x = 0 of the two reflected pulses. (c) Evaluate the time interval between the arrivals as the meteorite of the two outgoing pulses, as measured in the rest frame of the meteorite.

Answer (b) and (c) first with a well-chosen Lorentz transformation. Then answer again, this time using the Doppler effect and the results of the above problem (French 5-9).

A: (a) This spacetime graph of the meteorite was constructed in the program "Spacetime" by Prof. Edwin F. Taylor.



(b) Chose the frame of the Earth. The world line of the first pulse is

q[t, -t, 0,0]. The worldline of the meteorite is q[t, v t/c - d,0,0]. Solve for the time when the distances are the same.

```
Solve[
   q[t, β t - d, 0, 0][[2, 1]] ==
  q[t, -t, 0, 0][[2, 1]], t]
```

$$\left\{\left\{t \to \frac{d}{1+\beta}\right\}\right\}$$

The distance traveled is the same, so it arrives back at the Earth at

$$\text{pulse1}_{\text{back}} = \frac{2\,d}{1+\beta};$$

Find the time at which the second pulse arrives at the meteorite. The only change is the departure of the pulse.

```
pulse2time_met = Solve[
   q[t, β t - d, 0, 0][[2, 1]] ==
  q[t, -t + to, 0, 0][[2, 1]], t]
```

$$\left\{\left\{t \to -\frac{-d-\text{to}}{1+\beta}\right\}\right\}$$

Use the time to find pulse 2's position.

```
pulse2pos_met = β t - L /. pulse2time_met
```

$$\left\{-L - \frac{(-d-\text{to})\,\beta}{1+\beta}\right\}$$

Add these together to find the return time.

$$\text{pulse2}_{\text{back}} = \text{Simplify}\left[\frac{d+\text{to}}{1+\beta} + d - \frac{\beta(d+\text{to})}{1+\beta}\right]$$

$$\frac{2\,d + \text{to} - \text{to}\,\beta}{1+\beta}$$

Examine the interval between the arrival of the two pulses back to the Earth.

```
Simplify[pulse2_back - pulse1_back]
```

$$\frac{\text{to} - \text{to}\,\beta}{1+\beta}$$

The interval between the arrival of the two pulses is shifted by a factor $t' = \frac{1-\beta}{1+\beta} t_o$.

(c) Choose the rest frame of the meteorite. Boost the emission event to this frame.

```
pulse1source = q[γ[β], β γ[β], 0, 0] . q[0, d, 0, 0];
```

Add the time after being boosted together with the position needed to travel to the meteorite to get the time of pulse 1 at the meteorite.

```
pulse1met = pulse1source[[1, 1]] + pulse1source[[2, 1]]
```

$$\frac{d}{\sqrt{1-\beta^2}} - \frac{d\,\beta}{\sqrt{1-\beta^2}}$$

Repeat this process for pulse 2.

```
pulse2source = Simplify[L[to, d, β] . q[to, d, 0, 0]];
```

```
pulse2met = pulse2source[[1, 1]] + pulse2source[[2, 1]]
```

$$\frac{to - d\,\beta}{\sqrt{1-\beta^2}} + \frac{d - to\,\beta}{\sqrt{1-\beta^2}}$$

Examine the interval between the arrival of the two pulses at the meteorite.

```
Simplify[pulse2met - pulse1met]
```

$$\frac{to - to\,\beta}{\sqrt{1-\beta^2}}$$

The interval between the arrival of the two pulses at the meteorite is shifted by a factor $t' = \sqrt{\frac{1-\beta}{1+\beta}}\, t_o$.

(b') From the reference frame of the meteorite, the pulse of light would be blueshifted from the source, and blueshifted to the receiver. Use the result from 5-9 (b).

```
Simplify[(γ[-β] - β γ[-β])^2 q[to, to, 0, 0] . {1, 0, 0, 0}]
```

$$\left\{-\frac{to\,(-1+\beta)}{1+\beta}, -\frac{to\,(-1+\beta)}{1+\beta}, 0, 0\right\}$$

The time interval between pulses is $t' = \frac{1-\beta}{1+\beta} t_o$.

(c') As stated above, the pulse of light from the source is blueshifted, so using a modified answer from 5-9 (a).

```
Simplify[(γ[-β] - β γ[-β]) q[to, to, 0, 0] . {1, 0, 0, 0}]
```

$$\left\{-\frac{to\,(-1+\beta)}{\sqrt{1-\beta^2}}, -\frac{to\,(-1+\beta)}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

The time interval between pulse at the meteorite is $t' = \sqrt{\frac{1-\beta}{1+\beta}}\, t_o$.

French: 5-11

Q: An astronaut moves radially away from the Earth at a constant acceleration (as measured in the Earth's reference frame) of 9.8 m/s^2. How long will it be before the redshift makes the red glare of the neon signs of Earth invisible to his human eyesight?

A: Solve for the velocity which redshifts the wavelength of neon

(~600 nm) to invisible (~700 nm).

```
Solve[
  (γ[β] + β γ[β]) q[600, 600, 0, 0] [[2, 1]] == 700, β]
```

$$\left\{\left\{\beta \to \frac{13}{85}\right\}\right\}$$

The time required at a constant acceleration is t = v/a.

$$\frac{13\,c}{85}\,\frac{1}{9.8\,m/s^2}\,\frac{min}{60\,s}\,\frac{hr}{60\,min}\,\frac{day}{24\,hr}$$

54.1883 day

After 54 days, the neon lights become invisible to the astronaut's eyes.

French: 5-12

Q: There is a spaceship shuttle service from the Earth to Mars. Each spaceship is equipped with two identical lights, one at the front and one at the rear. The spaceships normally travel at a speed vo, relative to the Earth, such that the headlight of a spaceship approaching Earth appears green (500 nm) and the taillight of a departing spaceship appears red (600 nm). (a) what is the value of vo/c? (b) One spaceship accelerates to overtake the spaceship ahead of it. At what speed must the overtaking spaceship travel relative to the Earth so that the taillight of the Mars-bound spaceship ahead of it looks like a headlight (500 nm green)?

A: (a) Solve for the velocity that reverses the shifts to the same wavelength (i.e., redshift the headlight's blueshifted light to the wavelength of the taillight's blueshifted redshifted light ;)

```
Solve[
  (γ[β] + β γ[β]) q[500 nm, 500 nm, 0, 0][[2, 1]] ==
 (γ[β] - β γ[β]) q[600 nm, 600 nm, 0, 0][[2, 1]], β]
```

$$\left\{\left\{\beta \to \frac{1}{11}\right\}\right\}$$

The spaceships travel at vo/c = 1/11.

(b) Solve for the velocity needed to shift the wavelength from 600 to 500 nm.

```
Solve[
  (γ[β] - β γ[β]) q[600 nm, 600 nm, 0, 0][[2, 1]] == 500 nm, β]
```

$$\left\{\left\{\beta \to \frac{11}{61}\right\}\right\}$$

The required velocity is v/c = 11/61.

R & H: 2-68 A Doppler shift revealed as a color change

Q: A spaceship is receding from the Earth at a speed of 0.20c. A light on the rear of the ship appears blue (450 nm) to the passengers on the ship. What color would it appear to an observer on the Earth?

A: Redshift the light at 450 nm by 0.20c.

```
(γ[0.2] + 0.2 γ[0.2]) q[450 nm, 450 nm, 0, 0] . {1, 0, 0, 0}
{551.135 nm, 551.135 nm, 0, 0}
```

The light appears at 551 nm, yellow.

R & H: 2-71 The Ives-Stillwell experiment

Q: (long) Neutral hydrogen atoms are moving along the axis of an evacuated tube with a speed of 2.0 x 10ˆ6 m/s. A spectrometer is arranged to receive light emitted by these atoms in the direction of their forward motion. This light, if emitted from resting hydrogen atoms, would have a measured (proper) wavelength of 486.13 nm. (a) Calculate the expected wavelength for light emitted from the forward-moving (approaching) atoms, using the exact relativistic formula. (b) By using a mirror this same spectrometer can also measure the wavelength of light emitted by these moving atoms in the direction opposite to their motion. What wavelength is expected under this arrangement? (c) Calculate the difference between the average of the two wavelengths found in (a) and (b) and the unshifted (proper) wavelength. Find the second order dependence on beta. By this technique, Ives and Stillwell were able to distinguish between the predictions of the classical and the relativistic Doppler formulas.

A: (a) & (b) Red- and blue shift the light at 486 nm +/- the speed of the moving hydrogen.

```
βH = 2 10^6 m/s/c
1/150
(γ[βH] + βH γ[βH]) q[486.133, 486.133, 0, 0] . {1, 0, 0, 0}
{489.385, 489.385, 0., 0.}
(γ[βH] - βH γ[βH]) q[486.133, 486.133, 0, 0] . {1, 0, 0, 0}
{482.903, 482.903, 0., 0.}
```

The light is redshifted to 489.4 nm and blueshifted to 482.9 nm.

(c) We can measure the average of these two shifted wavelengths, or their average difference.

```
(489.385 + 482.903)/2 - 486.133
0.011
(489.385 - 482.903)/2
3.241
```

According to classical theory, if the observer is fixed, and the source moves, there is no second order dependence on beta.

$$\lambda = \lambda_o (1 - \beta)$$

If the source is fixed but the observer moves, then

$$\lambda = \lambda_o (1 - \beta + \beta^2)$$

Special relativity predicts a coefficient of +0.5 for the beta squared term, the one measured in the lab.

R&H: 2-83 The headlight effect

Q: A source of light, at rest in the S' frame, emits uniformly in all directions. The source is viewed from frame S, the relative speed parameter relating the two frames being beta. (a) Show that at high speeds, the forward-pointing cone into which the source emits half of its radiation has a half angle given closely, in radian measure, by

$$\theta_{1/2} = \sqrt{2(1-\beta)}$$

(b) What value of the half angle is predicted for the gamma radiation emitted by a beam of energetic neutral pions, for which v/c = 0.993? (c) At what speed would a light source have to move toward an observer to have half of its radiation concentrated into a narrow forward cone of half angle 5.0?

A: This problem requires a boost quaternion that works with nonzero values for t, x, y, and z. See the last problem set in the addition of velocities section for the derivation of the following boost quaternion, or the notebook on "Alternative algebra for boosts":

```
Λ[t_, x_, y_, z_, β_] :=
1/(t^2 + x^2 + y^2 + z^2) *
q[γ[β] t^2 - 2 β γ[β] t x + γ[β] x^2 + y^2 + z^2,
-β γ[β] (t^2 - x^2),
t y - x z + γ[β] (-t + β x) y + γ[β] (-β t + x) z,
x y + t z + γ[β] (β t - x) y + γ[β] (-t + β x) z]
```

Boost a spherically symmetric velocity quaternion, normalizing to the resulting gamma so the resulting quaternion still characterizes velocities.

```
βsphere = Simplify[
   Λ[1, 1, 1, 1, -β] . q[1, 1, 1, 1] . {1, 0, 0, 0} /
  (Λ[1, 1, 1, 1, -β] . q[1, 1, 1, 1])[[1, 1]]]
```

$$\left\{1, 1, \frac{\sqrt{1-\beta^2}}{1+\beta}, \frac{\sqrt{1-\beta^2}}{1+\beta}\right\}$$

Calculate the angle directly from the ratio of speeds.

```
θ0.5 = (βsphere[[3]] + βsphere[[4]])/βsphere[[2]]
```

$$\frac{2\sqrt{1-\beta^2}}{1+\beta}$$

As beta approaches 1, this angle approaches $\sqrt{2(1-\beta)}$.

(b) Let beta -> 0.993.

```
θ0.5 180/π /. β -> 0.993
6.79122
```

The predicted half angle for the gamma rays is 6.79.

(c) Solve for beta, given the angle.

```
Solve[θ0.5 == 5. π/180, β]
{{β → 0.9962}}
```

A light source would have to travel at 0.9962c to concentrate its radiation in a forward cone of half angle 5.

Four-Vector Invariants

Baranger: Decay of a particle - timelike or spacelike?

Q: In event 1, an unstable particle is produced in the target of an accelerator. In event 2, this particle decays 5 meters away. Is the interval between these two events timelike or spacelike? Why?

A: The speed of the particle must be less than one, or x/t < 1. If event 1 is at the origin and event 2 has a spatial position of 5m, it must have a time of 5m + a (a>0). Calculate the square of the interval by squaring the quaternion.

```
Simplify[(q[a + 5, 5, 0, 0] . q[a + 5, 5, 0, 0])[[1, 1]]]
a (10 + a)
```

The square of the interval aˆ2 + 10a is always positive, so the interval is timelike in the future.

R&H: 2-42 The interval is invariant - check it out

Q: Two events occur on the x axis of reference frame S, their spacetime coordinates being event1 = q[5 us, 720 m,0,0] and event 2 = [2 us, 1200 m,0,0]. (a) What is the square of the spacetime interval for these two events? (b) What are the coordinates of these events in a frame S’ that moves at speed 0.60c in the direction of increasing x? Calculate the square of the interval in this frame and compare it to the value calculated for frame S. (c) What are the coordinates of these events in a frame S” that moves at a speed of 0.95c in the direction of decreasing x? Again calculate the square of the spacetime interval and compare it with the values found in (a) and (b). Do your calculations bear out the invariance of the spacetime interval?

A: (a) The square of the spacetime interval between events 1 and 2 is the first term of difference between the quaternions squared.

```
event1 = q[5. 10^-6 s c, 720 m, 0, 0];

event2 = q[2 10^-6 s c, 1200 m, 0, 0];

((event1 - event2) . (event1 - event2))[[1, 1]]
579600. m^2
```

The square of the interval between event 1 and 2 is 5.8 x 10ˆ5 mˆ2.

(b) Boost the quaternions and then square them.

```
e1b6 = L[5. 10^-6 s c, 720 m, 0.6] . event1;

e2b6 = L[2. 10^-6 s c, 1200 m, 0.6] . event2;

((e1b6 - e2b6) . (e1b6 - e2b6))[[1, 1]]
579600. m^2
```

The square of the interval between the boosted events is the same.

(c) Repeat the exercise with a new value for beta.

```
e1b95 = L[5. 10^-6 s c, 720 m, -0.95] . event1;

e2b95 = L[2. 10^-6 s c, 1200 m, -0.95] . event2;

((e1b95 - e2b95) . (e1b95 - e2b95))[[1, 1]]
579600. m^2
```

Again, the square of the interval between the boosted events is the same. The first term of the square of a quaternion is identical to the first term of a square of a boosted quaternion.

R&H: 2-43 An event pair - timelike or spacelike?

Q: Two events occur on the x axis of reference frame S, their spacetime coordinates being event1 = q[5 us, 200 m,0,0] and event 2 = [2 us, 1200 m,0,0]. (a) What is the square of the spacetime interval for these two events? (b) What is the proper distance interval between them? (c) If two events possess a (mathematically real) proper distance interval, it should be possible to find a frame S' in which these events would be seen to occur simultaneously. Find this frame. (d) Can you calculate a (mathematically real) proper time interval for this pair of events? (e) Would you describe this pair of events as timelike? Spacelike? Lightlike?

A: (a) The square of the spacetime interval between events 1 and 2 is the first term of difference between the quaternions squared.

```
event1 = q[5. 10^-6 s c,  200 m, 0, 0];

event2 = q[2. 10^-6 s c, 1200 m, 0, 0];

((event2 - event1).(event2 - event1))[[1, 1]]
-190000. m^2
```

The square of the interval between event 1 and 2 is -1.9 x 10ˆ5 mˆ2

(b) The proper distance interval is the square root of the negative of this number.

$\sqrt{1.9\ 10^5\ m^2}$

$435.89 \sqrt{m^2}$

The proper distance is 436 m.

(c) Boost both event quaternions by beta, set the time components equal to each other, and solve for beta.

```
Solve[
  (L[5. 10^-6 s c,  200 m, β] . event1)[[1, 1]] ==
  (L[2. 10^-6 s c, 1200 m, β] . event2)[[1, 1]], β]
{{β → -0.9}}
```

In frame S', the events will appear simultaneous for v/c = 0.9 in the direction of decreasing x.

(d) & (e) For events that are spacelike separated, there is no meaningful measure of proper time.

R&H: 2-44 An event pair - spacelike or timelike?

Q: Two events occur on the x axis of reference frame S, their spacetime coordinates being event1 = q[5 us, 720 m,0,0] and event 2 = [2 us, 1200 m,0,0]. (a) Using the data from problem 2-42 above, calculate the proper time interval for this pair of events. The proper time interval that you have calculated should be smaller than any of the actual time intervals in the three given frames of problem 2-42. Is it? (b) If two events possess a (mathematically real) proper time interval, it should be possible to find a frame S' in which these events would be seen to occur at the same place. Find this frame. (c) Can you calculate a (mathematically real) proper distance interval for this pair of events? (d) Would you describe this pair of events as timelike? Spacelike? Lightlike?

A: (a) To make this question more of a challenge, let's define a quaternion "Ltau" which maps an arbitrary timelike quaternion to its proper time:

Ltau . q[t,x,y,z] = q[tau, 0, 0, 0] .

To find Ltau, multiply the above equation on the right by the inverse of q[t,x,y,z].

```
Simplify[
q[√((q[t, x, y, z] . q[t, x, y, z])[[1, 1]]), 0, 0, 0] .
    Transpose[q[t, x, y, z]] . {1, 0, 0, 0} /
    (t^2 + x^2 + y^2 + z^2)]
```

$$\left\{\frac{t\sqrt{t^2-x^2-y^2-z^2}}{t^2+x^2+y^2+z^2}, -\frac{x\sqrt{t^2-x^2-y^2-z^2}}{t^2+x^2+y^2+z^2}, -\frac{y\sqrt{t^2-x^2-y^2-z^2}}{t^2+x^2+y^2+z^2}, -\frac{z\sqrt{t^2-x^2-y^2-z^2}}{t^2+x^2+y^2+z^2}\right\}$$

```
Lτ[tlq_] :=

  Transpose[tlq] *
  √(tlq[[1, 1]]^2 - tlq[[2, 1]]^2 - tlq[[3, 1]]^2 - tlq[[4, 1]]^2) /
   (tlq[[1, 1]]^2 + tlq[[2, 1]]^2 + tlq[[3, 1]]^2 + tlq[[4, 1]]^2)
Lτ[q[32., 2., 5, -4.3]] . q[32, 2, 5, -4.3] . {1, 0, 0, 0}
{31.2492, 0., -1.0842 × 10^-19, 0.}
```

Works to within default accuracy.

Now on to the question. Map the given quaternion to its proper time interval.

```
intq = q[2. 10^-6 s, 1200 m/c, 0, 0] - q [5. 10^-6 s, 720 m/c, 0, 0];
(Lτ[intq] . intq)[[1, 1]]
2.53772 × 10^-6 √(s^2)
```

The proper time is 2.54 microseconds. This is less than the time of 3 microseconds observed in this frame.

Boost the interval up 0.6c, & repeat the cycle.

```
intqb6 =
  L[intq[[1, 1]], intq[[2, 1]], 0.6] . intq;
(Lτ[intqb6] . intqb6)[[1, 1]]
2.53772 × 10^-6 Sqrt[s^2]
intqb6[[1, 1]]
-4.95 × 10^-6 s
```

The interval 2.53 microseconds is the same, less that 4.95 microseconds observed.

```
intqb95 =
  L[intq[[1, 1]], intq[[2, 1]], -0.95] . intq;
(Lτ[intqb95] . intqb95)[[1, 1]]
2.53772 × 10^-6 Sqrt[s^2]
intqb95[[1, 1]]
-4.73979 × 10^-6 s
```

The interval is the same, less that 4.74 microseconds observed.

(b) Boost both event quaternions by beta, set the space components equal to each other, and solve for beta.

```
Solve[
  (L[5. 10^-6 s c,  720 m, β] . q[5. 10^-6 s c,  720 m, 0, 0])[[2, 1]] ==
  (L[2. 10^-6 s c, 1200 m, β] . q[2. 10^-6 s c, 1200 m, 0, 0])[[2, 1]], β]
{{β → -0.533333}}
```

The frame must move a speed 0.53c in the direction of decreasing x.

(c) & (d) The interval is timelike. It is not meaningful to search for a proper distance between these two events.

The Twin Paradox

The tortoise & the hare

Q: The tortoise challenges the hare to a race in the woods. The hare laughs hysterically saying "Surely, M'am, you are not serious?" But the tortoise is serious; she gets on the course and starts running(?) right away. The course is a closed loop beginning and ending at the same tree. While the tortoise is running, the hare continues telling jokes with his friends. But when he sees that she has almost gotten back to the finish, he decides that it is time to teach her a lesson, and he dashes on the course as quick as he can to catch up with her. Alas, he miscalculated slightly and he returns to the tree just barely behind her!

QUESTION: Assuming that the two animals were of the same age before the race, which one is older at the end of it? Justify your answer with quantitative arguments!

A: Let the hare run the fraction f of the tortoise's proper time t. Calculate the tortoise's squared interval in terms of this fraction.

```
τtort =
 (q[f t, 0, 0, 0] . q[f t, 0, 0, 0] +
   q[(1 - f) t, 0, 0, 0] . q[(1 - f) t, 0, 0, 0])[[1, 1]]
```

$(1-f)^2 t^2 + f^2 t^2$

In the tortoise's reference frame, the hare initially travels away from the tortoise at the slow Btort speed for time t. Then the hare starts traveling toward the tortoise at Bhare speed for a time (1-f) t. Calculate the hare's squared interval.

```
τhare =
  (q[f t, βtort f t, 0, 0] . q[f t, βtort f t, 0, 0] +
   q[(1 - f) t, -βhare f t, 0, 0] . q[(1 - f) t, -βhare f t, 0, 0])[[
   1, 1]]
```

$f^2 t^2 + (t - f t)^2 - f^2 t^2 \beta_{hare}^2 - f^2 t^2 \beta_{tort}^2$

Look at the difference.

```
Simplify[τhare - τtort]
```

$-f^2 t^2 (\beta_{hare}^2 + \beta_{tort}^2)$

Since this term is always negative, the hare is necessarily younger than the tortoise.

R&H: B2-2 Einstein and the clock "paradox"

Q: Einstein, in his first paper on the special theory of relativity, wrote the following: "If one of the two synchronous clocks at A is moved in a closed curve with constant velocity until it returns to A, the journey lasting t seconds, then by the clock that has remained at rest the travelled clock on its arrival at A will be t vˆ2/2 cˆ2 seconds slow." Prove this statement. (Note: Elsewhere in his paper Einstein indicated that this result is an approximation, valid only for v << c.)

A: Compare the intervals of the two clocks, one that has move, the other that has remained.

```
√((q[t, β t, 0, 0] . q[t, β t, 0, 0])[[1, 1]]) - t
```

$-t + \sqrt{t^2 - t^2 \beta^2}$

If beta << 1, calculate the series expansion.

```
Simplify[PowerExpand[
   Series[%, {β, 0, 2}]]]
```

$-\frac{t \beta^2}{2} + O[\beta]^3$

The moving clock is t vˆ2/2 cˆ2 slower than the one at rest.

R&H B2-12: Getting Younger

Q: Can you think of any way to use space travel to reverse the aging process, that is, to get younger? Could you send your parents out on a long space voyage and have them be younger than you are when they get back?

A: There are actually two questions here. Starting with the last question first, with a HUGE investment of energy for the parents, time will appear to run at a slower rate than the clocks back at home. The energy investment is the critical parameter to determine if the clocks will run at different enough rates to have the parents return younger than their children.

The second question concerns reversing the aging process. The aging process will appear to procede in the same manor for both parent and child. Why is this not reversable? Find the quaternion that reverses time.

LTimeRev q[t, x, y, z] = q[-t, x, y, z]

Compute LTimeRev by multiplying on the right by the inverse of q[t,x,y,z].

```
Simplify[q[-t, x, y, z] .
 Transpose[q[t, x, y, z]] . {1, 0, 0, 0}/
    (t^2 + x^2 + y^2 + z^2)]
```

$$\left\{\frac{-t^2+x^2+y^2+z^2}{t^2+x^2+y^2+z^2}, \frac{2\,t\,x}{t^2+x^2+y^2+z^2}, \frac{2\,t\,y}{t^2+x^2+y^2+z^2}, \frac{2\,t\,z}{t^2+x^2+y^2+z^2}\right\}$$

Aboard the spaceship, or on the Earth, t >>> x, y and z, so the time reversal quaternion is approximately

```
LTimeRevBsmall[t_, x_, y_, z_] :=

  q[-1, 2 x/t, 2 y/t, 2 z/t]
```

Test that this works for someone moving a meter per second in the x direction, 0.5 m/s in the y.

```
LTimeRevBsmall[1. s, 1. m/c, 0.5 m/c, 0] .

      q[1. s, 1. m/c, 0.5 m/c, 0] . {1, 0, 0, 0}
```

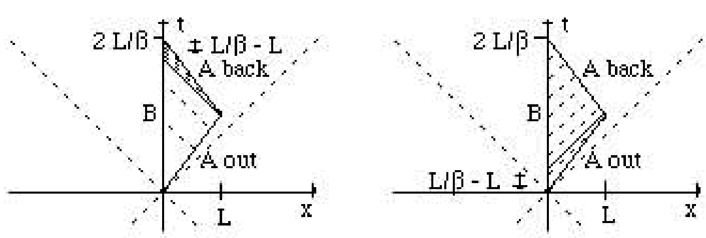
$\{-1.\ s, 3.33333\times 10^{-9}\ s, 1.66667\times 10^{-9}\ s, 0.\ s\}$

The proposed quaternion does reverse time in the classical regime. Note that it is predominantly a scalar, almost q[-1,0,0,0] However, it is not _exactly_ the identity. If we think about time reversal for two nearby worldlines, they will not commute by the small factor found in the second through fourth terms. This observation may lead to a new justification of the second law of thermodynamics.

French: 5-20 Signals from twins

Q: A and B are twins. A goes on a trip to Alpha Centauri (4 light-years away) and back again. He travels at speed 0.6c with respect to the Earth both ways, and transmits a radio signal every 0.01 year in his frame. His twin B similarly sends a signal every 0.01 years in his own rest frame. (a) how many signals emitted by A before he turns around does B receive? (b) How many signals does A receive before he turns around? (c) What is the total number of signals each twin receives from the other? (d) Who is younger at the end of the trip, and by how much? Show that the twins both agree on this result.

A: Start out by drawing the signals sent and received from B's frame of reference.



B's reference frame A's signals sent

B's reference frame B's signals sent

(a) The signals from A out are redshifted and received at B for a time of ...

```
time_BgetsAout = 2 4 yr / 0.6 - (4 yr / 0.6 - 4 yr)
10.6667 yr
```

The signals are sent at a rate of 100/year as viewed by the sender. This rate is lowered by the redshifting, so the total number of signals is the lower rate times the amount of time the signals are received.

```
(γ[0.6] - 0.6 γ[0.6]) q[100/yr, 100/yr, 0, 0] . {1, 0, 0, 0} tBgetsAout
{533.333, 533.333, 0, 0}
```

B receives 533 redshifted signals from A.

(b) The signals from B are not shifted in B's frame, but are received at A for a time of...

```
time_AgetBout = (4 yr / 0.6 - 4 yr)
2.66667 yr
```

```
q[100/yr, 100/yr, 0, 0] . {1, 0, 0, 0} 2.667 yr
{266.7, 266.7, 0, 0}
```

A receives 266 signals from B during A's trip to Alpha Centauri.

(c) It is easiest to calculate the number of signals received by A since the rate with B as a reference frame is constant.

```
A_total = q[100/yr, 100/yr, 0, 0] . {1, 0, 0, 0} 2 4 yr/0.6
{1333.33, 1333.33, 0, 0}
```

A receives a total of 1333 signals from B.

B gets blueshifted signals for a short time. The rate of signals goes way up.

```
time_BgetAback = (4 yr / 0.6 - 4 yr)
2.66667 yr
```

```
(γ[0.6] + 0.6 γ[0.6]) q[100/yr, 100/yr, 0, 0] . {1, 0, 0, 0} 2.667 yr
{533.4, 533.4, 0, 0}
```

```
B_total = 2 * 533.3
1066.6
```

B receives a total of 1066 signals from A.

(d) B has sent out 1333 signals, but has only received 1066 from A, so he has experienced more ticks of the clock. The difference is due to the instantaneous change of reference frame experienced by A.

Post Ramble

There are a few tools required to solve problems in special relativity using quaternions to characterize events in spacetime. The most basic are a round value for c and gamma.

$$c = 3\ 10^8\ m/s;$$

$$\gamma[\beta\_] := \frac{1}{\sqrt{1 - \beta^2}}$$

Define a function for quaternions using its matrix representation.

$$q[t\_, x\_, y\_, z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

A quaternion L that transforms a quaternion (L q[**x**] = q[**x'**]) identical to how the Lorentz transformation acts on 4-vectors

(Lambda **x** = **x'**) should exist. These are described in detail in the notebook "A different algebra for boosts." For boosts along the x axis with y = z = 0, the general function for L is

```
L[t_, x_, β_] :=
 1/(t^2 + x^2) q[γ[β] t^2 - 2 γ[β] β t x + γ[β] x^2, -β γ[β] (t^2 - x^2), 0, 0]
```

Most of the problems here involve much simpler cases for L, where t or x is zero, or t is equal to x.

If t = 0, then

```
L[0, x, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If x = 0, then

```
L[t, 0, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, -\frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If t = x, then

```
Simplify[L[t, t, β] . {1, 0, 0, 0}]
```

$$\left\{\frac{1-\beta}{\sqrt{1-\beta^2}}, 0, 0, 0\right\}$$

Note: this is for blueshifts. Redshifts have a plus instead of the minus.

7 8.033 Problem Set 5: Energy, momentum, and mass

French: 6-3

Q: A particle of rest mass m and kinetic energy 2 m c^2 strikes and sticks to a stationary particle of rest mass 2 m. Find the rest mass M of the composite particle.

A: Calculate the square of the mass of the system by squaring the quaternion and looking at the first term.

$$\left(q\left[(1.\,m + 2\,m) + 2\,m, \sqrt{\left(\frac{3}{1}\right)^2 - 1}\,1\,m, 0, 0\right] . \; q\left[(1.\,m + 2\,m) + 2\,m, \sqrt{\left(\frac{3}{1}\right)^2 - 1}\,1\,m, 0, 0\right]\right)[[1, 1]]$$

$17.\,m^2$

The mass of the composite particle is $\sqrt{17}\,m$.

French: 6-4

Q: (a) A photon of energy E collides with a stationary particle of rest mass mo and is absorbed. What is the velocity of the resulting composite particle? (b) A particle of rest mass mo moving at a speed of 4/5c collides with a similar particle at rest and forms a composite particle. What is the rest mass of the composite particle and what is its speed?

A: (a) The speed can be calculated from the ratio of the momentum to the energy. Define the before and after quaternions, set these ratios equal to each other and solve.

$$\text{beforeq} = q[m\,C^2 + e_\lambda, e_\lambda, 0, 0];$$

$$\text{afterq} = q[\gamma\,M\,C^2, \gamma\,\beta\,M\,C^2, 0, 0];$$

$$\text{Solve}\left[\frac{\text{beforeq}[[2,1]]}{\text{beforeq}[[1,1]]} == \frac{\text{afterq}[[2,1]]}{\text{afterq}[[1,1]]}, \beta\right]$$

$$\left\{\left\{\beta \to \frac{e_\lambda}{C^2\,m + e_\lambda}\right\}\right\}$$

The velocity of the composite particle is $\beta = \frac{e}{e + m\,c^2}$.

(b) The problem is the same, only the numbers have been changed to protect the writer.

$$\text{beforeq} = q[m + 5/3\,m, 4/3\,m, 0, 0];$$

$$\text{Solve}\left[\frac{\text{beforeq}[[2,1]]}{\text{beforeq}[[1,1]]} == \frac{\text{afterq}[[2,1]]}{\text{afterq}[[1,1]]}, \beta\right]$$

$$\left\{\left\{\beta \to \frac{1}{2}\right\}\right\}$$

Because mass is conserved, the mass of the composite equals the mass of the system before the collision.

$$\sqrt{(\text{beforeq} . \text{beforeq})[[1,1]]}$$

$$\frac{4\sqrt{m^2}}{\sqrt{3}}$$

The composite travels at 0.5c with a mass of $\frac{4}{\sqrt{3}}\,m$.

The projectile

Q: A projectile of rest mass M1, energy E1, and momentum p1, is directed at a stationary target of mass m2. Find a simple expression for the velocity of the center of mass frame.

A: Define the before and after quaternions (a boost to the center of mass frame - not a collision!).

```
beforeq = q[γ1 m1 + m2, γ1 β1 m1, 0, 0];
```

```
afterq = q[M_CM, 0, 0, 0];
```

We need to find a boost that will transform between the two. The boost quaternion is simple for the center of mass frame where x=y=z=0.

```
afterq_boosted = q[γ_cm, -γ_cm β_cm, 0, 0] . afterq;
```

The ratio of the second component to the first one is p/E or v/c, exactly what we are looking for.

$$\text{Solve}\left[\frac{\text{afterq}_{\text{boosted}}[[2,1]]}{\text{afterq}_{\text{boosted}}[[1,1]]} == \frac{\text{beforeq}[[2,1]]}{\text{beforeq}[[1,1]]}, \beta_{cm}\right]$$

$$\left\{\left\{\beta_{cm} \to -\frac{m1\,\beta_1\,\gamma_1}{m2 + m1\,\gamma_1}\right\}\right\}$$

The velocity of the center of mass frame is $\beta_{CM} = \frac{-\beta\gamma m_1}{m_2 + \gamma m_1}$.

French: 6-11

Q: The neutral pi meson decays into two gamma rays (and nothing else). If the pion (whose rest mass is 135 MeV) is moving with a kinetic energy of 1 GeV: (a) What are the energies of the gamma rays if the decay process causes them to be emitted in opposite directions along the pion's original line of motion? (b) What angle is formed between the two gamma rays if they are emitted at equal angles to the direction of the pion's motion?

A: (a) Define the before and after quaternions.

$$\text{beforeq} = q\left[1135.\,\text{MeV}, \sqrt{\left(\frac{1135.}{135.}\right)^2 - 1}\,135\,\text{MeV}, 0, 0\right];$$

```
afterq = q[E1 + E2, E1 - E2, 0, 0];
```

Solve for E2 using energy conservation, then E1 using momentum conservation.

```
Solve[beforeq[[2, 1]] - afterq[[2, 1]] == 0 /.
(Solve[beforeq[[1, 1]] - afterq[[1, 1]] == 0, E2])[[1, 1]], E1]
{{E1 → 1130.97 MeV}}
```

One gamma is 1131 MeV, the other is 4 MeV.

(b) The after quaternion has been changed. Solve first for the energy, then the angle.

```
afterq = q[2 e_ray, 2 e_ray Cos[θ],
                    e_ray Sin[θ] +
     e_ray Sin[θ + π], 0];
```

```
Solve[beforeq[[2, 1]] - afterq[[2, 1]] == 0 /.
(Solve[beforeq[[1, 1]] - afterq[[1, 1]] == 0, e_ray])[[1, 1]], θ]
{{θ → -0.119225}, {θ → 0.119225}}
```

Theta is the half angle in radians.

$$0.1192\ 2\ \frac{180}{\pi}$$

```
13.6593
```

There is 13.6 between the two gamma rays.

French: 6-14

Q: Show that the following processes are dynamically impossible: (a) A single photon strikes a stationary electron and gives up all its energy to the electron. (b) A single photon in empty space is transformed into an electron and a positron. (c) A fast positron and a stationary electron annihilate, producing only one photon.

A: (a) By inspection, momentum is conserved only if E = 0.

```
beforeq = q[eλ + me C^2, eλ, 0, 0];

afterq = q[M C^2, 0, 0, 0];

Solve[beforeq[[2, 1]] - afterq[[2, 1]] == 0, eλ]
{{eλ → 0}}
```

A photon with no momentum is no photon at all.

(b) The square of the mass of the photon is zero. Examine the same for the electron and positron.

```
afterq = q[Cosh[θe] m + Cosh[θe] m,
                Sinh[θe] m - Sinh[θe] m,
    0, 0];

Simplify[(afterq . afterq)[[1, 1]]]
2 (1 + Cosh[θe + θe]) m^2
```

The mass in never less than 2 m, so this transition violates conservation of mass.

(c) The mass of the photon is zero. Find out the mass of the electron and positron..

```
beforeq = q[Cosh[θe] m + m, Sinh[θe] m, 0, 0];

Simplify[(beforeq . beforeq)[[1, 1]]]
2 m^2 (1 + Cosh[θe])
```

There is no choice of theta that makes the mass zero, so the transformation is not possible without a violation of conservation of mass.

French: 7-1

Q: A K meson traveling through the laboratory breaks up into two pi mesons. One of the pi mesons is left at rest. What was the energy of the K? What is the energy of the remaining pi meson? (Rest mass of K meson = 494 MeV; rest mass of pi meson ≈137 MeV).

A: Define the quaternions for the before and after states.

```
beforeq = q[γk 494 MeV, Sqrt[γk^2 - 1] 494 MeV, 0, 0];

afterq = q[γπ 137 MeV + 137 MeV, Sqrt[γπ^2 - 1] 137 MeV, 0, 0];

Solve[(afterq . afterq)[[1, 1]] == (494. MeV)^2, γπ]
{{γπ → 5.50104}}
```

The kinetic energy will be E - M = gamma M - M.

```
(5.501 - 1) 137 MeV
616.637 MeV
```

The pion has 616.6 MeV of kinetic energy.

Use the fact that energy is conserved to calculate the kinetic energy of the K meson.

```
(afterq[[1, 1]] /. γπ -> 5.501) - 494 MeV
396.637 MeV
```

The K meson has 396.6 MeV of kinetic energy.

Baranger: Protons to K's

Q: Consider the annihilation of an antiproton with a proton, both particles being at rest, according to the reaction

$$p + \bar{p} \Rightarrow K^o + (\bar{K})^o$$

mass p = 940 MeV/cˆ2, mass *K* = 500 MeV/cˆ2. (a) Find the kinetic energies and momenta of the created *K* particles. (b) If the proper lifetime of the *K*'s is 10ˆ − 10 s, find their actual lifetime in the lab frame and the distance they travel.

A: (a) The mass of the system is 1880 MeV. After the creation of the *K*'s, the total momentum is still zero.

```
beforeq = q[1880 MeV, 0, 0, 0];

afterq = q[2 γ 500. MeV, γ β 500 MeV - γ β 500 MeV, 0, 0];

Solve[beforeq[[1, 1]] - afterq[[1, 1]] == 0, γ]
{{γ → 1.88}}
```

$KE_{k^o} = (1.88 - 1)\ 500\ MeV$

440. MeV

$P_{k^o} = \sqrt{1.88^2 - 1}\ 500\ MeV$

795.99 MeV

Each *K* has 440 MeV of kinetic energy and a momentum of 796 Mev.

(b) t' = gamma t and d' = beta c t'. Simple stuff.

$t_{lab} = 1.88\ 10^{-10} s$

1.88×10^{-10} s

$d_{lab} = \sqrt{\frac{1.88^2 - 1}{1.88^2}}\ c\ t_{lab}$

0.0477594 m

In 0.188 nanoseconds, the *K*'s travel 4.77 cm.

Baranger: Photon collision

Q: A photon of energy E = 600 MeV, traveling in the +x direction, hits a stationary particle of rest energy M cˆ2 = 1000 MeV. After the collision, these two particles are replaced by two new particles of masses m1 and m2 respectively. The total energy (including rest energy) of particle 1 is E1 = 700 MeV and its momentum is p1x = 400 MeV, p1y = 300 MeV. (a) Draw a "before and after" picture. (b) Find the energy and the momentum of particle 2. (c) Find the masses m1 and m2.

A: (b) Lots of data is given to define the quaternions.

```
beforeq = q[600. MeV, 600 MeV, 0, 0] + q[1000 MeV, 0, 0, 0];
particle1 = q[700. MeV, 400 MeV, 300 MeV, 0];
particle2 = q[e2, p2x, p2y, 0];
afterq = particle1 + particle;
```

Because energy and mass are conserved, we know by inspection that for particle 2, E2 = 900 MeV, p2x c = 200 MeV, p2y c = -300 MeV.

```
(particle2 = beforeq - particle1) . {1, 0, 0, 0}
{900. MeV, 200 MeV, -300 MeV, 0}
```

(c) Calculate the masses by squaring the quaternions.

$$\sqrt{(\text{particle1 . particle1})[[1,1]]}$$

$$489.898\sqrt{\text{MeV}^2}$$

$$\sqrt{(\text{particle2 . particle2})[[1,1]]}$$

$$824.621\sqrt{\text{MeV}^2}$$

The masses are m1 = 490 MeV and m2 = 825 Mev.

Baranger: Multiplying 4 vectors

Q: Let the energy-momentum 4-vector be

$$\vec{p} = (e,\ p\,c)$$

Let the 4-velocity be the 4-vector

$$\vec{u} = \gamma\,(1,\ v/c)$$

Prove (carefully, rigorously!) that, if a particle of 4 momentum p is observed by an observer of 4-velocity u, then the energy of the particle in the frame of the observer is

$$\vec{p}\,.\,\vec{u} = p_t\,u_t - p\,.\,u$$

HINT: Choose special, convenient coordinate axes for the space components (and say what these axes are).

A: Choose to define the velocity vector in terms of the momentum vectors. Define the two quaternions.

$$p_4 = q[e_t,\ p_x,\ p_y,\ p_z];$$

$$\left(u_4 = \frac{p_4}{\sqrt{(p_4\,.\,p_4)[[1,1]]}\,e_t}\right).\{1,0,0,0\}$$

$$\left\{\frac{1}{\sqrt{e_t^2-p_x^2-p_y^2-p_z^2}},\ \frac{p_x}{e_t\sqrt{e_t^2-p_x^2-p_y^2-p_z^2}},\ \frac{p_y}{e_t\sqrt{e_t^2-p_x^2-p_y^2-p_z^2}},\ \frac{p_z}{e_t\sqrt{e_t^2-p_x^2-p_y^2-p_z^2}}\right\}$$

Calculate their product, looking at the first term which has units of energy.

$$(p_4\,.\,u_4)[[1,1]]$$

$$\frac{e_t}{\sqrt{e_t^2 - p_x^2 - p_y^2 - p_z^2}} - \frac{p_x^2}{e_t \sqrt{e_t^2 - p_x^2 - p_y^2 - p_z^2}}$$
$$- \frac{p_y^2}{e_t \sqrt{e_t^2 - p_x^2 - p_y^2 - p_z^2}} - \frac{p_z^2}{e_t \sqrt{e_t^2 - p_x^2 - p_y^2 - p_z^2}}$$

The result can be written as $p_t\, u_t\; - \;p\; .\; u$.

There is an even easier way to express this product.

```
Simplify[%]
```

$$\frac{\sqrt{e_t^2 - p_x^2 - p_y^2 - p_z^2}}{e_t}$$

This is M/E or 1/gamma. This result was due to the unusual choice of the velocity vector. Look at the other terms.

```
Simplify[(p4 . u4)[[2, 1]]]
```

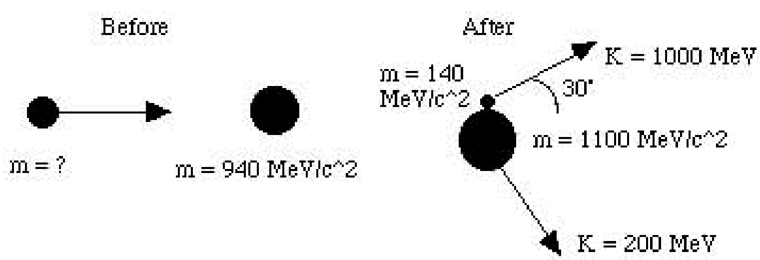
$$\frac{2\, p_x}{\sqrt{e_t^2 - p_x^2 - p_y^2 - p_z^2}}$$

The entire quaternion product is

$$\vec{p}\,\vec{u} = 1/\gamma + 2\gamma\beta_x I + 2\gamma\beta_y J + 2\gamma\beta_z K$$

Baranger: An inelastic collision

Q: Consider the following inelastic collision:



Find the mass of the incoming projectile.

A: Determine the gammas from the kinetic energy and masses given.

$$\gamma_{140} = \frac{140.\,\text{MeV} + 1000\,\text{MeV}}{140\,\text{MeV}}$$

8.14286

$$\gamma_{1100} = \frac{1100.\,\text{MeV} + 200\,\text{MeV}}{1100\,\text{MeV}}$$

1.18182

We can calculate the py momentum for the m140 particle.

$$py_{140} = \sqrt{(\gamma_{140}{}^2 - 1)}\ \text{Sin}\left[30\,\frac{\pi}{180}\right] 140\,\text{MeV}$$

565.685 MeV

The py momentum for the m1100 must be equal and opposite, which allows a calculation of the angle.

$$\text{ArcSin}\left[\frac{py_{140}}{\sqrt{\gamma_{1100}{}^2 - 1}\,1100\,\text{MeV}}\right] \frac{180}{\pi}$$

```
54.7356
```

Calculate the amount of momentum in the x direction.

$$px_{140} = \sqrt{(\gamma_{140}{}^2 - 1)}\ \text{Cos}\left[30\ \frac{\pi}{180}\right]\ 140\ \text{MeV}$$

```
979.796 MeV
```

$$px_{1100} = \sqrt{(\gamma_{1100}{}^2 - 1)}\ \text{Cos}\left[54.736\ \frac{\pi}{180}\right]\ 1100\ \text{MeV}$$

```
399.996 MeV
```

$$px_{tot} = px_{140} + px_{1100}$$

```
1379.79 MeV
```

All of this momentum is from the incoming projectile.

Calculate the total energy after minus m940 to get the energy of the incoming projectile.

```
140 MeV + 1000 MeV + 1100 MeV + 200 MeV - 940 MeV
1500 MeV
```

Calculate the mass of incoming projectile the usual way by looking at the first term of the square root of the squared quaternion.

$$\sqrt{(q[1500\ \text{MeV},\ px_{tot},\ 0,\ 0]\ .\ q[1500\ \text{MeV},\ px_{tot},\ 0,\ 0])[[1,1]]}$$

$$588.365\ \sqrt{\text{MeV}^2}$$

The mass of the incoming projectile is 588 MeV.

Post Ramble

There are a few tools required to solve problems in special relativity using quaternions to characterize events in spacetime. The most basic are a round value for c and gamma.

$$c = 3\ 10^8\ \text{m/s};$$

$$\gamma[\beta\_] := \frac{1}{\sqrt{1 - \beta^2}}$$

Define a function for quaternions using its matrix representation.

$$q[t\_,\ x\_,\ y\_,\ z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

A quaternion L that transforms a quaternion (L q[**x**] = q[**x'**]) identical to how the Lorentz transformation acts on 4-vectors

(Lambda **x** = **x'**) should exist. These are described in detail in the notebook "A different algebra for boosts." For boosts along the x axis with y = z = 0, the general function for L is

$$L[t\_,\ x\_,\ \beta\_] := \frac{1}{t^2 + x^2}\ q[\gamma[\beta]\ t^2 - 2\,\gamma[\beta]\,\beta\, t\, x + \gamma[\beta]\ x^2,\ -\beta\,\gamma[\beta]\ (t^2 - x^2),\ 0,\ 0]$$

Most of the problems here involve much simpler cases for L, where t or x is zero, or t is equal to x.

If t = 0, then

```
L[0, x, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If x = 0, then

```
L[t, 0, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, -\frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If t = x, then

```
Simplify[L[t, t, β] . {1, 0, 0, 0}]
```

$$\left\{\frac{1-\beta}{\sqrt{1-\beta^2}}, 0, 0, 0\right\}$$

Note: this is for redshifts. Blueshifts have a plus instead of the minus.

8 8.033 Problem Set 6: The Compton effect and threshold collision problems

French: 6-16 The Compton effect

Q: The usual theory of the Compton effect considers a stationary free electron being struck by a photon, resulting in a scattered photon of lower energy. Suppose that a photon (of energy E) has a head-on collision with a _moving_ electron (of rest mass mo) What initial velocity does the electron have if the collision results in a photon recoiling straight backward with the same energy E as the incident photon?

A: There are the same number of particles before and after, traveling in opposite directions with the same energy. Define the before and after quaternions.

```
beforeq = q[eλ + γ[β] mo C², eλ - βγ[-β] mo C², 0, 0];

afterq  = q[eλ + γ[β] mo C², -eλ + βγ[β] mo C², 0, 0];
```

Conserve momentum and solve for the speed.

```
Simplify[Solve[
   beforeq[[2, 1]] - afterq[[2, 1]] == 0, β]]
```

$$\left\{\left\{\beta \to -\frac{e_\lambda}{\sqrt{e_\lambda^2 + C^4 m_o^2}}\right\}, \left\{\beta \to \frac{e_\lambda}{\sqrt{e_\lambda^2 + C^4 m_o^2}}\right\}\right\}$$

The initial velocity of the electron is $\beta = \frac{e}{\sqrt{e^2 + m^2 c^4}}$.

French: 6-17

Q: A stream of very high energy photons (>>10 MeV) is fired at a block of matter. Show that the energy E of the photons scattered directly backward is essentially independent of the energy of the incident photons. What is the value of E backward?

A: Define the relevant quaternions.

```
beforeq = q[e_forward + m C², e_forward, 0, 0];

afterq = q[e_back + γ[β] m C², -e_back + βγ[β] m C², 0, 0];
```

Solve for the energy forward using energy conservation.

```
Simplify[Solve[
   beforeq[[1, 1]] - afterq[[1, 1]] == 0, e_forward]]
```

$$\left\{\left\{e_{forward} \to C^2 m \left(-1 + \frac{1}{\sqrt{1-\beta^2}}\right) + e_{back}\right\}\right\}$$

Solve for the energy back using momentum conservation and the previous result.

```
Simplify[Solve[
   beforeq[[2, 1]] - afterq[[2, 1]] == 0 /.
    %[[1]], e_back]]
```

$$\left\{\left\{e_{back} \to \frac{C^2 m \left(-1 + \beta + \sqrt{1-\beta^2}\right)}{2\sqrt{1-\beta^2}}\right\}\right\}$$

As beta approaches one, the value of E backward approaches 1/2 mcˆ2. The photons are most likely to collide with electrons of mass 511 MeV, so E backward is approximately 255 MeV.

French: 6-18

Q: (a) A photon of energy hv collides elastically with an electron at rest. After the collision the energy of the photon is hv/2, and it travels in a direction making an angle of 60 with its original direction. What is the value of the frequency? What sort of photon is it? (b) A photon of energy hv collides with an excited atom at rest. After the collision the photon still has energy hv, but its direction has changed by 180. If the atom is in its ground state after the collision, what was its initial excitation energy?

A: (a) Define the before and after quaternions.

```
beforeq = q[hν + m C^2, hν, 0, 0];

afterq = q[hν/2 + Cosh[α] m C^2, hν/4 + Cos[θ] Sinh[α] m C^2,
                    √3hν/4 +
     Sin[θ] Sinh[α] m C^2, 0];
```

By conservation of energy and momentum, the beforeq minus the afterq should be zero.

```
(consq = beforeq - afterq) . {1, 0, 0, 0}
```

$$\left\{\frac{h\nu}{2} + C^2 m - C^2 m \,\mathrm{Cosh}[\alpha],\ \frac{3\,h\nu}{4} - C^2 m \,\mathrm{Cos}[\theta]\,\mathrm{Sinh}[\alpha],\ -\frac{\sqrt{3}\,h\nu}{4} - C^2 m \,\mathrm{Sin}[\theta]\,\mathrm{Sinh}[\alpha],\ 0\right\}$$

There are 3 equations (each term is equal to zero) and three unknowns (hv, alpha, and theta). Eliminate theta using a trig identity, then eliminate alpha using a hyperbolic trig identity. (It's not easy, but it does work).

```
elimθ =
 (Cos[θ] /. Solve[consq[[2, 1]] == 0, Cos[θ] ])[[1]]^2 +
  (Sin[θ] /. Solve[consq[[3, 1]] == 0, Sin[θ] ])[[1]]^2
```

$$\frac{3\,h\nu^2\,\mathrm{Csch}[\alpha]^2}{4\,C^4\,m^2}$$

```
elimα =
  (Cosh[α] /. Solve[consq[[1, 1]] == 0, Cosh[α] ])[[1]]^2 -
  (Sinh[α] /. Solve[1/elimθ == 1, Sinh[α] ])[[1]]^2
```

$$-\frac{3\,h\nu^2}{4\,C^4\,m^2} + \frac{(-h\nu - 2\,C^2\,m)^2}{4\,C^4\,m^2}$$

This identity equals 1, so it can be solved for hv.

```
Solve[elimα == 1, hν]
```

$$\{\{h\nu \to 0\},\ \{h\nu \to 2\,C^2\,m\}\}$$

Find the frequency by putting in the appropriate constants.

$$\frac{2\ 9.1\ 10^{-31}\ \mathrm{kg}\ c^2}{6.62\ 10^{-34}\ \mathrm{kg}\ m^2/s}$$

$$\frac{2.47432\times 10^{20}}{s}$$

The frequency of the photon is 2.5 10^20 s^-1, a gamma ray.

(b) Define the before and after quaternions.

```
beforeq = q[hν + m* C^2, hν, 0, 0];

afterq = q[hν + γ[β] m C^2, -hν + β γ[β] m C^2, 0, 0];
```

Solve for beta in terms of the ground state using momentum conservation.

```
Solve[beforeq[[2, 1]] - afterq[[2, 1]] == 0, β]
```

$$\left\{\left\{\beta \to -\frac{2\,h\nu}{\sqrt{4\,h\nu^2 + C^4\,m^2}}\right\}, \left\{\beta \to \frac{2\,h\nu}{\sqrt{4\,h\nu^2 + C^4\,m^2}}\right\}\right\}$$

Solve for the excited state using energy conservation.

```
Solve[beforeq[[1, 1]] - afterq[[1, 1]] == 0 /. %[[1]], m*]
```

$$\left\{\left\{m^* \to \frac{m}{\sqrt{\frac{C^4\,m^2}{4\,h\nu^2 + C^4\,m^2}}}\right\}\right\}$$

The excitation energy will be

$$m_{ex}\,c^2 = m^*c^2 - m\,c^2 = m\,c^2\left(\sqrt{\left(\left(\frac{2\,h\nu}{m\,c^2}\right)^2 + 1\right)} - 1\right).$$

French: 6-19

Q: A high-energy photon strikes and is scattered by a proton that is initially stationary and completely free to recoil. The proton is observed to recoil at a 30 angle with a kinetic energy of 100 MeV. (a) What was the energy of the incident photon? (b) What are the direction and energy of the scattered photon?

A: (a) Define the before and after quaternions.

$$\text{beforeq} = q[e_\lambda + 100\,\text{MeV} + 938\,\text{MeV},\ e_\lambda + 100\text{MeV},\ 0,\ 0];$$

$$\text{afterq} = q\left[e_\lambda + 1039\,\text{MeV},\ e_\lambda\,\text{Cos}[\theta] + \sqrt{\left(\frac{1038\,\text{MeV}}{938\,\text{MeV}}\right)^2 - 1}\,\text{Cos}\left[30\,\frac{\pi}{180.}\right]\,938\,\text{MeV},\ e_\lambda\,\text{Sin}[\theta] + \sqrt{\left(\frac{1038\,\text{MeV}}{938\,\text{MeV}}\right)^2 - 1}\,\text{Sin}\left[30\,\frac{\pi}{180.}\right]\,938\,\text{MeV},\ 0\right];$$

By conservation of energy and momentum, the beforeq minus the afterq should be zero.

```
(consq = beforeq - afterq) . {1, 0, 0, 0}
```

$$\{-\text{MeV},\ -284.968\,\text{MeV} + e_\lambda - \text{Cos}[\theta]\,e_\lambda,\ -222.261\,\text{MeV} - \text{Sin}[\theta]\,e_\lambda,\ 0\}$$

There are 2 equations (each term is equal to zero) and 2 unknowns (E and theta). Eliminate the angle using a trig identity.

```
elimθ =
 (Cos[θ] /. Solve[consq[[2, 1]] == 0, Cos[θ]])[[1]]^2 +
  (Sin[θ] /. Solve[consq[[3, 1]] == 0, Sin[θ]])[[1]]^2
```

$$\frac{49400.\,\text{MeV}^2}{e_\lambda^2} + \frac{1.\,(-284.968\,\text{MeV} + e_\lambda)^2}{e_\lambda^2}$$

This identity equals 1, so it can be solved for hv.

```
Solve[elimθ == 1, eλ]
```

$$\{\{e_\lambda \to 229.16\,\text{MeV}\}\}$$

The incident photon is E + 100 MeV, or 329 MeV.

(b) The energy of the scattered photon was calculated above at 229 MeV. The momentum in the conservation quaternion can be solved for theta.

```
Solve[consq[[2, 1]] == 0 /. eλ- > 229 MeV, θ]
{{θ → -1.8177}, {θ → 1.8177}}
1.8177 180/π
104.147
```

The photon scatters at an angle of 104.

French: 7-2

Q: An electron-positron pair can be produced by a gamma ray striking a stationary electron:

$$\gamma + e^- \rightarrow e^- + e^+ + e^-$$

What is the minimum gamma ray energy that will make this process go?

A: Define the before and after quaternions.

```
beforeq = q[eγ, eγ, 0, 0] + q[me, 0, 0, 0];

afterq = q[3 me, 0, 0, 0];
```

Set the square of the masses equal to each other and solve for E.

```
Solve[
  (beforeq . beforeq)[[1, 1]] -
  (afterq . afterq)[[1, 1]] == 0, eγ]
{{eγ → 4 me}}

% /. me - > 511 MeV
{{eγ → 2044 MeV}}
```

The minimum energy of the gamma ray is 2044 MeV.

Baranger: Threshold KE

Q: What is the threshold kinetic energy, in the Lab system, for the reaction:

$$p + p \rightarrow p + p + \pi + \pi$$

The protons have mass M, the pions have mass m. Set c = 1.

A: K = E - M. gamma = E/M = K/M + 1. These relations can be used to define the before and after quaternions.

$$\text{beforeq} = q\left[2M + K, \sqrt{\left(\frac{K}{M} + 1\right)^2 - 1}\, M, 0, 0\right];$$

```
afterq = q[2 M + 2 m, 0, 0, 0];
```

Set the masses equal to each other and solve for K.

```
Solve[
  (beforeq . beforeq)[[1, 1]] -
  (afterq . afterq)[[1, 1]] == 0, K]
```

$$\left\{\left\{K \rightarrow \frac{2\,(m^2 + 2\,m\,M)}{M}\right\}\right\}$$

The minimum kinetic energy required is K = (2mˆ2 + 4m M)/M.

French: 7-3

Q: Suppose that a certain accelerator can give protons a kinetic energy of 200 GeV. The rest mass mo of a proton is 0.938 GeV. Calculate the largest possible rest mass Mo of a particle X that could be produced by the impact of one of these high-energy protons on a stationary proton in the following process:

p + p → p + p + X

A: K = E - M. gamma = E/M = K/M + 1. These relations can be used to define the before and after quaternions.

```
beforeq = q[2 0.938 GeV + 200 GeV,
   Sqrt[(200/0.938 + 1)^2 - 1] 0.938 GeV, 0, 0];
afterq = q[2 0.938 GeV + X, 0, 0, 0];
```

$$\text{beforeq} = q\left[2\ 0.938\,\text{GeV} + 200\,\text{GeV},\ \sqrt{\left(\frac{200}{0.938}+1\right)^2 - 1}\ 0.938\,\text{GeV},\ 0,\ 0\right];$$

Set the square of the masses equal to each other and solve for X.

```
Solve[
  (beforeq . beforeq)[[1, 1]] -
  (afterq . afterq)[[1, 1]] == 0, X]
{{X → -21.3367 GeV}, {X → 17.5847 GeV}}
```

The largest possible rest mass is 17.6 GeV for this accelerator.

Baranger: Another threshold

Consider the reaction:

p + π → π + p + π

The target proton is at rest. The rest mass of the proton is 940 MeV and the pion is 140 MeV. (a) Calculate the threshold kinetic energy of the incident pion for this reaction. (b) At threshold, calculate the velocities and momenta of the final-state proton and the pions in the lab system.

A: (a) Define the before and after quaternions.

$$\text{beforeq} = q\left[940.\,\text{MeV} + (140\,\text{MeV} + K),\ \sqrt{\left(\frac{K}{140\,\text{MeV}}+1\right)^2 - 1}\ 140\,\text{MeV},\ 0,\ 0\right];$$

```
afterq = q[2 140 MeV + 940 MeV, 0, 0, 0];
```

Set the square of the masses equal to each other and solve for K.

```
Solve[
  (beforeq . beforeq)[[1, 1]] -
  (afterq . afterq)[[1, 1]] == 0, K]
{{K → 171.277 MeV}}
```

The threshold kinetic energy of the pion is 171 MeV.

(b) In the center-of-mass frame, all three particles are at rest. In the Lab frame, all three particles have the same velocity, but different momenta. Calculate the energy before.

```
beforeE = 940 MeV + 140 MeV + 171 MeV
1251 MeV
```

The gamma after is this energy divided by the mass.

$$\frac{\mathbf{1251.\,MeV}}{\mathbf{2\,140\,MeV + 940\,MeV}}$$

1.02541

Knowing gamma, both velocity and momenta can be calculated.

$$\sqrt{\frac{1.0254^2 - 1}{1.0254^2}}$$

0.221197

$$p_{pion} = \sqrt{1.0254^2 - 1}\;140\,\text{MeV}$$

31.7541 MeV

$$p_{proton} = \sqrt{1.0254^2 - 1}\;940\,\text{MeV}$$

213.206 MeV

The particles have a relativistic velocity of 0.22, the pions have a momentum pc = 31.7 MeV, and the protons have a momentum of pc= 213 MeV.

French: 7-6

Q: The kinetic energy K of a system in the lab frame is related to the kinetic energy K* in the center of mass frame in the nonrelativistic case by the expression K = K* + MVˆ2/2, where M is the total mass of the system and V is the velocity of the center of mass. What is the analogous expression for the relativistic case? Show that it reduces to the above result if all speeds are much less than c.

A: Boost the quaternion for the center-of-mass frame to the lab, where Mo is the sum of the rest masses of the particles in the system.

```
qlab = q[γ[-β], -β γ[-β], 0, 0].q[Mo c^2 + KCM, 0, 0, 0];
```

Energy is conserved, so $e_{lab} = K_{lab} + M_o c^2$.

```
Solve[qlab[[1, 1]] == Klab + Mo C^2, Klab]
```

$$\left\{\left\{K_{lab} \to -C^2 M_o + \frac{K_{CM} + C^2 M_o}{\sqrt{1-\beta^2}}\right\}\right\}$$

The kinetic energy in the lab frame is $K_{lab} = \gamma(K_{CM} + M_o c^2) - M_o c^2$.

Look at the nonrelativistic limit.

```
Series[γ[β] (KCM + Mo C^2) - Mo C^2, {β, 0, 2}]
```

$$K_{CM} + \frac{1}{2}(K_{CM} + C^2 M_o)\,\beta^2 + O[\beta]^3$$

Note that (K + Mo cˆ2) is the total mass of the system. In the nonrelativistic limit, the lab kinetic energy equals the kinetic energy within the center of mass frame plus the kinetic energy of all the individual masses in the center of mass frame.

Post ramble: Initialization functions

There are a few tools required to solve problems in special relativity using quaternions to characterize events in spacetime. The most basic are a round value for c and gamma.

```
c = 3 10^8 m/s;
```

$$\gamma[\beta\_] := \frac{1}{\sqrt{1-\beta^2}}$$

Define a function for quaternions using its matrix representation.

$$q[t\_, x\_, y\_, z\_] := \begin{pmatrix} t & -x & -y & -z \\ x & t & -z & y \\ y & z & t & -x \\ z & -y & x & t \end{pmatrix}$$

A quaternion L that transforms a quaternion (L q[**x**] = q[**x'**]) identical to how the Lorentz transformation acts on 4-vectors

(Lambda **x** = **x'**) should exist. These are described in detail in the notebook "A different algebra for boosts." For boosts along the x axis with y = z = 0, the general function for L is

$$L[t\_, x\_, \beta\_] := \frac{1}{t^2 + x^2} q[\gamma[\beta]\, t^2 - 2\gamma[\beta]\beta\, t\, x + \gamma[\beta]\, x^2, -\beta\gamma[\beta]\,(t^2 - x^2), 0, 0]$$

Most of the problems here involve much simpler cases for L, where t or x is zero, or t is equal to x.

If t = 0, then

```
L[0, x, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, \frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If x = 0, then

```
L[t, 0, β] . {1, 0, 0, 0}
```

$$\left\{\frac{1}{\sqrt{1-\beta^2}}, -\frac{\beta}{\sqrt{1-\beta^2}}, 0, 0\right\}$$

If t = x, then

```
Simplify[L[t, t, β] . {1, 0, 0, 0}]
```

$$\left\{\frac{1-\beta}{\sqrt{1-\beta^2}}, 0, 0, 0\right\}$$

Note: this is for blueshifts. Redshifts have a plus instead of the minus.

The problems are from "Basic Concepts in Relativity" by Resnick and Halliday, ©1992 by Macmillian Publishing, "Special Relativity" by A. P. French, © 1966, 1968 by MIT, and Prof. M. Baranger of MIT.