LXXVI. Quaternionic Form of Relativity.

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It has been remarked by Cayley,[1] as early as in 1854, that the rotations in a four-dimensional space may be effected by means of a pair of quaternions applied, one as a prefactor and the other as a postfactor, to the quaternion q whose components are the four coordinates of a space-point, say

$$q' = aqb, \tag{1}$$

where in the case of *pure* rotation a and b must of course be either *unit*-quaternions or at least such that $\mathcal{T}^2 a . \mathcal{T}^2 b = 1$; $\mathcal{T}$ denoting the tensor.

On the other hand, it is widely known that the so-called Lorentz-transformation of the union of ordinary space (x, y, z) and time (t), which is the basis of the modern theory of Relativity, corresponds precisely to a (hyperbolic) rotation of the four-dimensional manifoldness (x, y, z, t), or of what Minkowski called the "world."

Hence the obvious idea of representing explicitly the Lorentz-transformation in the quaternionic shape (1),— which, together with some allied questions, will be the subject of the present paper.

To solve this simple problem we have only to write down the well-known relativistic transformation, *i.e.*, the formulæ of Einstein, then to develop the triple product in (1) and to compare the two.

For our purpose it will be most convenient to put Einstein's formulæ at once in vector form, eliminating thus the quite unessential choice of the axes of coordinates. Let the vector $\mathbf{v} = v\mathbf{u}$ denote the uniform velocity of the system $S'(x', y', z', t')$

[1] A. Cayley Phil. Mag. vol. vii. (1854), and *Journ. f. reine u. angew. Mathem.* vol. 50 (1855); or 'Papers,' vol. ii. Cayley limited himself to the elliptic, *i.e.* real, rotations, but the extension to the hyperbolic and parabolic cases was an obvious matter. For the whole subsequent literature of the subject, see the article of E. Study in the *Encyclopédie d. Sc. Math.*, tome i. vol. i. fascicule 3, p. 452; Paris and Leipzig, 1908. See also F. Klein and A. Sommerfeld's work *Ueber d. Theorie des Kreisels*, iv. pp. 939-943; Leipzig, 1910. It was in fact a general hint at Relativity made by these authors on p. 942 that, after I had a whole year tried in vain a great variety of quaternionic operations for relativistic purposes, suggested to me the choice of the particular form (1).

relatively to the system $S(x,y,z,t)$.[2] Let O, O' be a pair of points in S and S', respectively, which coincide with one another for $t = t' = 0$. Call $\mathbf{r}$ $(= x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ the vector drawn in S from O as origin, and $\mathbf{r}'$ the corresponding vector in S', drawn from O' as origin. Then the transformation in question may be stated as follows:— The component of $\mathbf{r}'$ normal to the velocity $\mathbf{v}$ is equal to that of $\mathbf{r}$, *i.e.*

$$\mathbf{r}' - \left(\mathbf{r}'\mathbf{u}\right)\mathbf{u} = \mathbf{r} - (\mathbf{r}\mathbf{u})\,\mathbf{u}, \tag{α}$$

whilst the component of $\mathbf{r}'$ taken along the direction of motion is altered according to the formula

$$\mathbf{r}'\mathbf{u} = \gamma[(\mathbf{r}\mathbf{u}) - vt], \tag{β}$$

where $\gamma = \left(1-\beta^2\right)^{-1/2}$, $\beta = v/c < 1$, c = velocity of light.[3] Finally, the time is transformed according to

$$t' = \gamma\left[t - \frac{1}{c^2}(\mathbf{r}\mathbf{v})\right].$$

To get the resultant $\mathbf{r}'$ take the sum of (α) and of $(\beta)\times\mathbf{u}$. Then write, for the sake of subsequent convenience,

$$\ell = \imath ct, \qquad \imath = \sqrt{-1},$$

and similarly $\ell' = \imath ct'$.

Thus, the relativistic formulæ will become

$$\left.\begin{aligned} \mathbf{r}' &= \mathbf{r} + (\gamma-1)(\mathbf{r}\mathbf{u})\mathbf{u} + \imath\beta\gamma\ell\mathbf{u} \\ \ell' &= \gamma[\ell - \imath\beta(\mathbf{r}\mathbf{u})], \end{aligned}\right\} \tag{2}$$

quite independent of any system of coordinate-axes.

Now, to obtain the required quaternionic representation (1) of the whole transformation (2), let us introduce the quaternion

$$q = \mathbf{r} + \ell = \mathbf{r} + \imath ct, \tag{3}$$

and similarly

$$q' = \mathbf{r}' + \ell' = \mathbf{r}' + \imath ct'. \tag{3'}$$

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[2] $\mathbf{u}$ being a unit-vector in the direction of motion of S' relatively to S and v the absolute magnitude of its velocity.

[3] In these and in all following formulæ $(\mathbf{r}\mathbf{u})$, generally $(\mathbf{AB})$, means the modern scalar product of the vectors $\mathbf{A}$, $\mathbf{B}$, that is to say $AB\cos(\mathbf{A},\mathbf{B})$; hence $(\mathbf{AB})$ is the *negative* scalar part of the complete Hamiltonian product, $\mathbf{AB}$:

$$(\mathbf{AB}) = -\mathcal{S}\,\mathbf{AB}.$$

On the other hand, the modern vector product $\mathcal{V}\mathbf{AB}$ is *identical* with Hamilton's $\mathcal{V}\mathbf{AB}$.

Then the problem will consist in finding a pair of quaternions a, b such that

$$\mathbf{r}' + \ell' = a(\mathbf{r} + \ell)b,$$

and will be solved by developing the right side of this equation.

Having done this, explicitly, and compared with (2), I found immediately that the quaternions a, b can differ from one another only by an ordinary scalar factor, and since this may be distributed equally among a, b (their tensors entering only by the product), we may as well take simply *equal* a, b, say, both $= Q$. In fact, then, the form (1) is much too general for our purpose. Thus, to spare the reader any superfluous complication, let us at once seek for

$$q' = QqQ \tag{1a}$$

as the quaternionic equivalent of (2).

Denote the unknown vector and scalar parts of Q by $\mathbf{w}$ and s respectively, *i.e.* write

$$Q = \mathbf{w} + s \tag{4}$$

Then, developing the complete product of q, Q, by (3) and (4), and by the fundamental rules of Hamilton's Calculus,

$$qQ = \mathcal{V}\mathbf{rw} + \ell\mathbf{w} + s\mathbf{r} - (\mathbf{rw}) + s\ell,$$

and similarly

$$\begin{aligned} q' = QqQ &= \mathcal{V}\mathbf{w}\,\mathcal{V}\mathbf{rw} - \mathbf{w}(\mathbf{rw}) + 2s\ell\mathbf{w} + s^2\mathbf{r} - 2s(\mathbf{wr}) + (s^2 - w^2)\ell \\ &= (w^2 + s^2)\mathbf{r} - 2(\mathbf{rw})\mathbf{w} + 2s\ell\mathbf{w} + (s^2 - w^2)\ell - 2s(\mathbf{rw}), \end{aligned}$$

whence, splitting into the vector and scalar parts,

$$\left.\begin{aligned} \mathbf{r}' &= \left(w^2 + s^2\right)\mathbf{r} - 2(\mathbf{rw})\mathbf{w} + 2s\ell\mathbf{w} \\ \ell' &= \left(s^2 - w^2\right)\ell - 2s(\mathbf{rw}) \end{aligned}\right\} \tag{5}$$

Comparing this with (2), we get at once, as the conditions to be fulfilled by $\mathbf{w}$, s,

$$\left.\begin{aligned} w^2 + s^2 = 1; \quad s^2 - w^2 = \gamma; \quad 2sw = \imath\beta\gamma \\ \mathbf{w} = w\mathbf{u}. \end{aligned}\right\} \tag{6}$$

Hence $w = \pm\sqrt{(1-\gamma)/2}$, $s = \pm\sqrt{(1+\gamma)/2}$, where, to satisfy the third of the conditions (6), we must take *both* square roots with the upper or both with the lower sign; therefore

$$Q = \pm\left(\sqrt{(1+\gamma)/2} + \mathbf{u}\sqrt{(1-\gamma)/2}\right),$$

and since in $(1a)$ the quaternion Q appears twice, the choice of the $\pm$ sign becomes indifferent.

Thus, we obtain finally *the required quaternionic expression of the relativistic transformation*

$$\left.\begin{aligned} &q' = QqQ \\ \textit{with}\qquad &Q = \frac{1}{\sqrt{2}}\left(\sqrt{1+\gamma}+\mathbf{u}\sqrt{1-\gamma}\right), \end{aligned}\right\} \qquad (I.)$$

$\mathbf{u}$ *being a unit vector in the direction of motion of* S' *relatively to* S.

Observe that $\gamma = (1-v^2/c^2)^{-1/2} > 1$, so that the vector of Q is *imaginary*, whilst its scalar is real.

The tensor of Q is 1; thus denoting its *angle* by α, *i.e.*, writing

$$Q = \cos\alpha + \mathbf{u}\sin\alpha = e^{\alpha\mathbf{u}}, \qquad (7)$$

we have, by $(I.)$,

$$\cos\alpha = \sqrt{(1+\gamma)/2}, \qquad \sin\alpha = \sqrt{(1-\gamma)/2}.$$

Hence

$$\sin 2\alpha = \sqrt{1-\gamma^2} = \imath\beta\gamma = \frac{\imath\beta}{\sqrt{1-\beta^2}}$$

or

$$2\alpha = \operatorname{arctg}(\imath\beta) = \operatorname{arctg}\left(\imath\frac{v}{c}\right). \qquad (8)$$

Now this is precisely the (imaginary) angle of rotation in the plane t, x,[4] of Minkowski's four-dimensional world, corresponding to the transformation (2). Hence, by $(I.)$ and (7), we may say that one half of this rotation is effected by Q as a prefactor and the other half by the same quaternion as a postfactor.[5] This circumstance throws a peculiar light on each of our Q's.

[4] The axis of x coinciding with $\mathbf{u}$, and x itself being our $(\mathbf{ru})$.

[5] At the first sight it might seem that, the axis of Q being $\mathbf{u}$, this quaternion turns $\mathbf{r}$ round $\mathbf{u}$, *i.e.* in the plane y, z normal to $\mathbf{u}$, while in Minkowski's representation the rotation is in the plane x, t. But this is only an apparent contradiction. In fact,

$$Q\mathbf{r} = \cos\alpha.\mathbf{r} + \sin\alpha.\, \mathcal{V}\mathbf{ur} + \text{scalar},$$

that is to say, Q as a prefactor turns the transversal component of $\mathbf{r}$ round $\mathbf{u}$ by the angle $+\alpha$ and stretches its longitudinal component; similarly Q as a postfactor, besides stretching the longitudinal component of $\mathbf{r}$, turns its transversal component round $\mathbf{u}$ by the angle $-\alpha$, thus undoing the rotatory effect of the prefactor. Hence, what remains in the final result is but a stretching of $\mathbf{r}$'s longitudinal component and a change of ℓ or t, and this amounts precisely to the Minkowskian rotation in the plane x, t.

But what we are mainly concerned with is their union, which considered as an operator may be written

$$\omega = Q[\quad]Q, \tag{I.a}$$

the vacant place being destined for the operand.

We have just seen that this simple operator converts the quaternion $q = \mathbf{r} + \iota ct$ into its relativistic correspondent q'. Our q is equivalent to Minkowski's "space-time-vector of the first kind" or to Sommerfeld's "Vierervektor" x, y, z, ℓ. These authors call by this same name any *such and only such* tetrad of scalars (three real and the fourth imaginary) which transforms in the same way as x, y, z, ℓ,—adding where it is necessary the emphasizing epithet "Weltvector"[6]. Similarly we could call our q and any covariant quaternion a "world-quaternion"; but possibly the less pretentious name *physical quaternion* will do as well. Also, at least in the beginning, no further specification of the "kind" is needed.

Thus $\omega = Q[\quad]Q$, defined by $(I.)$, or by (7) and (8), is what I should like to call the relativistic *transformer* of any physical quaternion.

To get the inverse transformer ω^{-1}, viz. that which turns q' into q, apply to both sides of the equation $q' = QqQ$ the inverse quaternion Q^{-1} as a pre- and a postfactor; then, remembering that $Q^{-1}Q = QQ^{-1} = 1$, the result will be

$$q = Q^{-1}q'Q^{-1},$$

or

$$\omega^{-1} = Q^{-1}[\quad]Q^{-1},$$

and since Q is a unit quaternion, its inverse is also its conjugate, *i.e.* Hamilton's $\mathcal{K}Q$, which may be more conveniently written Q_c; hence

$$\omega^{-1} = Q_c[\quad]Q_c, \tag{I.b}$$

where $Q_c = \cos\alpha - \mathbf{u}\sin\alpha$. Thus, we see that the inverse transformer is got from the direct simply by changing the sign of the angle α or by inverting the direction of $\mathbf{u}$,—as it must be.

[6]H. Minkowski, *Die Grundgleichungen für d. elektromagn. Vorgänge in bewegten Körpern*, Götting. Nachrichten, 1908; Raum und Zeit, *Physik. Zeitschrift*, vol. x. (1909), also separatim. A. Sommerfeld, *Zur Relativitätstheorie*, i. and ii., *Annalen d. Physik*, vol. xxxii., xxxiii. (1910).

See also the admirably clear and beautiful book *Das Relativitätsprinzip* by M. Laue (Braunschweig, 1911), where the whole work of Einstein, Minkowski, and Sommerfeld, together with the author's own contributions, will be found fully developed.

Observe that, since the product of quaternions is distributive, the transformer ω has also the *distributive* property, *i.e.*, A, B being any quaternions,[7]

$$Q\,[A+B]\,Q = QAQ + QBQ, \tag{9}$$

and consequently, if ∂ be any scalar differentiator, also

$$Q\,[\partial A]\,Q = \partial QAQ,$$

since Q, being constant, is not exposed to ∂'s action. Again, by the *associative* property of quaternionic products, the dot signifying a separator,

$$A.QBQ = AQ.BQ,$$

and so on. For our present purpose we scarcely need a full enumeration of ω's properties.

In the above we have been concerned with q as an example, or in fact the very prototype, of a physical quaternion. Another example, which will be needed in the sequel, is the quaternionic equivalent of Sommerfeld's "Viererdichte," or Laue's "Viererstrom," say

$$C = \rho\left(\imath + \frac{1}{c}\mathbf{p}\right), \tag{10}$$

which we may accordingly call the *current-quaternion*. Here ρ means the volume-density of electricity and $\mathbf{p}$ the velocity of its motion relatively to the system S. To prove that C is a *physical quaternion*, write $\mathbf{p} = d\mathbf{r}/dt$, and consequently

$$C = \imath\rho\frac{dq}{d\ell}, \tag{10a}$$

and notice that, the charges of corresponding volumes in S and S' being equal (by a fundamental postulate), $d\ell/\rho$ is itself an invariant of the Lorentz-transformation.

The transformer $(I.a)$ may, of course, be applied not only to quaternionic magnitudes, but also to operators, as, for example, to differentiators, which have the structure of a quaternion. If Ω be an operator of this kind, in the system S, and Ω' its relativistic correspondent in S', and if $\Omega' = Q\Omega Q$, we shall say that the operator Ω has the *character* of a physical quaternion.

As a chief example of such an operator, which also will be needed for what follows, we shall consider here our quaternionic equivalent of Minkowski's matrix

[7] *I.e.* generally complete quaternions but also, more especially, pure scalars or pure vectors; either simple- or bi-vectors, that is to say real or complex. The **heavy type** (and this merely to suit the general custom) shall be henceforth used only for *pure vectors*, both real and complex.

called by him “lor” to the honour of Lorentz. This will simply be the Hamiltonian ∇ plus the scalar differentiator $\partial/\partial\ell$. Let us denote it by $\mathcal{D}$,

$$\mathcal{D} = \frac{\partial}{\partial\ell} + \nabla \tag{11}$$

$$= \partial/\partial\ell + \mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z.$$

Comparing this with

$$q = \ell + \mathbf{r} = \ell + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z,$$

we see at once that the operator $\mathcal{D}$ will transform precisely as q did, *i.e.*

$$\mathcal{D}' = Q\mathcal{D}Q. \tag{12}$$

Thus $\mathcal{D}$ has the character of a physical quaternion.

To obtain the above representation of the relativistic formulæ (2) we have introduced the quaternion $q = \ell + \mathbf{r}$. Now, for this purpose we might as well have used its conjugate, *i.e.*,

$$q_c = \ell - \mathbf{r}, \tag{3a}$$

and the corresponding $q'_c = l' - \mathbf{r}'$.[8] It may often be convenient to recur to q_c and it is therefore of some interest to know how it transforms. Now, a glance at (2) suffices to see that both of these formulæ remain unchanged if, having changed the signs of $\mathbf{r}$, $\mathbf{r}'$ (and leaving ℓ, ℓ' as before), we change also the sign of $\mathbf{u}$. Thus it is seen that

$$q'_c = Q_c q_c Q_c, \qquad \text{say} \quad = \omega_c q_c \tag{I.c}$$

where

$$Q_c = \cos\alpha - \mathbf{u}\sin\alpha = e^{-\alpha\mathbf{u}}.$$

Now q_c has precisely the same office as q, that is to say, $(I.)$ and $(I.c)$ are but two expressions of one and the same thing, namely, of the Lorentz-transformation. Hence q_c and any quaternion covariant with q_c is certainly a physical quaternion as well as q and its covariants.

Thus, *the conjugate of a physical quaternion will again be a physical quaternion.* If the original transformed as q, its conjugate will transform as q_c. If A is

[8] It can be proved immediately that $(q_c)' = (q')_c$. Therefore both may be written simply q'_c.

Notice also that the invariance of q's tensor, $\mathcal{T}q' = \mathcal{T}q$, which follows immediately from $(I.)$ (since Q is a unit quaternion), may be written:

$$q'q'_c = qq_c.$$

covariant with q, then A_c is covariant with q_c, and *vice versa*. Speaking of a physical quaternion we shall, when necessary, add the explanation cov. q or cov. q_c. But generally, for the sake of shortness, this will be omitted, and any letters, as A, B, a, b, &c., *without the subscript* $_c$ will be used to denote quaternions *covariant with* q. Observe that, with the above (formal) extension of our original definition, two physical quaternions may be either covariant with one another or not; in the last case we may call them *antivariant*, one being cov. q, and the other cov. q_c. Thus, by the above convention, A, B_c, or a, b_c will denote pairs of antivariant quaternions, the first in each pair transforming as q, and the second as q_c.

The above transformer $\omega_c = Q_c[\quad]Q_c$, which by $(I.b)$ becomes simply identical with ω^{-1}, is, of course, distributive, quite in the same way as $\omega = Q[\quad]Q$. Thus the *sum*, or difference, of two *mutually covariant* (but not of antivariant) physical quaternions will again be a physical quaternion.

The *reciprocal* of a physical quaternion is also a physical quaternion. For we have

$$a^{-1} = a_c\,(\mathcal{T}a)^{-2},$$

while the tensor $\mathcal{T}a$ of a physical quaternion is already known to be an invariant. Notice that a and a^{-1} are mutually antivariant.

Now for the *product* of physical quaternions. Take any pair a, b of such quaternions. Leave aside ab which transforms in the unmanageable way $a'b' = QaQ^2bQ$ (a, b being torn asunder), and pass at once to the product of antivariant factors, which might perhaps be called the *alternating product*, say

$$L = a_c b. \tag{13}$$

Then $L' = Q_c a_c Q_c . QbQ$, whence by the associative property, and remembering that $Q_c Q = 1$,

$$L' = Q_c L Q. \tag{13$'$}$$

Thus, L is certainly not a physical quaternion of the kind already considered; but since it is transformed in such a simple way and since it has, as will be seen in the sequel, an almost immediate bearing upon relativistic Electromagnetism, it deserves to be considered a little more fully. Consider, then, the conjugate of L. Remember the elementary rule, by which the conjugate of the product of any number of quaternions is the product of their conjugates in the *reversed* order, *i.e.* in our case

$$L_c = b_c a. \tag{14}$$

Now, transforming this, we get in quite the same way as above

$$L'_c = Q_c L_c Q. \tag{14$'$}$$

Hence we see that

$$Q_c[\quad]Q \qquad (II.)$$

is the relativistic transformer of *both* $R = a_c b$ *and its conjugate* L_c. Similarly,

$$Q[\quad]Q_c \qquad (II.a)$$

will be the transformer of *both* $R = ab_c$ *and its conjugate* $R_c = ba_c$. Thus the behaviour of L and R is characteristically distinct from that of q or of q_c.

Without trying as yet to invent for these kinds of quaternions any particular names, let us provisionally call any quaternion which is transformed by $(II.)$ or by $(II.a)$ an *L-quaternion* and an *R-quaternion*, respectively.[9]

Now, $Q_c[\quad]Q$, being the transformer of *both* L and L_c, is also the transformer of their sum and of their difference, *i.e.* also of the *scalar* and of the *vector parts* of the quaternion L *separately*, $s = \mathcal{S}L$ and $\mathbf{A} = \mathcal{V}L$. Now, s being a scalar, we have

$$s' = Q_c s Q = s Q_c Q = s,$$

i.e. s is an invariant. Then

$$\mathbf{A}' = Q_c \mathbf{A} Q,$$

and since Q, Q_c are unit quaternions, the tensor of $\mathbf{A}$ is another invariant.

Thus, the scalar of any L-quaternion and the tensor of its vector are invariants, while the vector itself is transformed into

$$\mathcal{V}L' = Q_c\,[\mathcal{V}L]\,Q \qquad (III.)$$

Or use the form $L = \sigma(\cos\varepsilon + \mathbf{a}\sin\varepsilon)$, where $\mathbf{a}$ is the unit of $\mathbf{A}$. Then $\sigma\cos\varepsilon$ and $\sigma\sin\varepsilon$ are invariants and consequently also σ and ε, so that another form of the last theorem will be:—

The tensor and the angle (or argument) of any L-quaternion are invariants, while its axis is transformed by $Q_c[\quad]Q$.

In quite the same way it will be seen that $\mathcal{S}R$ is invariant and

$$\mathcal{V}R' = Q\,[\mathcal{V}R]\,Q_c, \qquad (III.a)$$

or in other words:—

The tensor and the angle of any R-quaternion are invariants, while its axis is transformed by $Q[\quad]Q_c$.

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[9] *L*, *R*, being initials of *left*, *right*, may remind us of the position of *that* of the two generating factors which (as a_c or b_c) has the subscript $_c$, *i.e.* which is cov. q_c.

If we wish to return to the generating factors a_c &c., we can write the above properties

$$\mathcal{S}\, a'_c b' = \mathcal{S}\, a_c b \tag{15}$$

$$\mathcal{V} a'_c b' = Q_c\, [\mathcal{V} a_c b]\, Q \tag{16}$$

and similarly

$$\mathcal{S}\, a' b'_c = \mathcal{S}\, a b_c \tag{15a}$$

$$\mathcal{V} a' b'_c = Q\, [\mathcal{V} a b_c]\, Q_c \tag{16a}$$

But as a rule it is better to avoid any splitting of quaternions, if we are to expect simplicity and other advantages from the use of quaternionic language.

Now take the product of any number of L-quaternions, say L_1, L_2, L_3 &c.; then we see by $(II.)$ that all the internal Q's and Q_c's as it were neutralize one another, and what is left is only the Q_c at the beginning and the Q at the end of the whole chain. That is to say *the product of any number of L-quaternions is again an L-quaternion*. In quite the same way we see, by $(II.a)$, that *the product of any number of R-quaternions is again an R-quaternion*.

Notice also that, a being any physical quaternion cov. q (not necessarily that implied in L or in R), aL and Ra *are again physical quaternions,*[10] *and so are also* La_c and $a_c R$, namely

$$aL \quad \text{and} \quad Ra \quad \text{cov.}\, q. \tag{$IV.$}$$

$$La_c \quad \text{and} \quad a_c R \quad \text{cov.}\, q_c. \tag{$IV.a$}$$

Thus, the alternating product of any number of physical quaternions ($ab_c de_c \ldots$) furnishes us *either an L- or R- quaternion or again* (biquaternions covariant with) *the primary physical quaternions*, and never anything more.[11]

One remark more before leaving this subject. Suppose we are given the equation

$$bX = a,$$

in which a, b are cov. q. What is the relativistic transformer of X? To get it, write the given equation $X = b^{-1}a$ and remember that b^{-1} is cov. q. Thus the transformer of X will be the same as for $b_c a$, i.e. $Q_c[\quad]Q$. In other words, X will be an L-quaternion,

$$X = b^{-1} a\, \text{cov.}\, L. \tag{17}$$

[10] Or more exactly *biquaternions* (in Hamilton's sense of the word) transforming like the primary physical quaternions. *Cf.* p. 808, *infra*.

[11] So much as to the *alternating* products. And as regards the products of *covariant* factors, like *ab*, I have not, up to the present, been able to make out any of their possible applications to physical subjects, and shall therefore not consider them here at all.

This will, of course, be still the case if we have instead of b the above differential operator $\mathcal{D}$, i.e.:

$$\text{if} \quad \mathcal{D}X = a, \quad \text{then} \quad X \quad \text{is} \quad \text{cov.}\, L, \tag{V.}$$

or the transformer of X is $Q_c[\quad]Q$. For $\mathcal{D}$ has the structure of q, and the entire manipulation with the Q's is done precisely as before, since Q, Q_c, being constant in space and time, are not exposed to $\mathcal{D}$'s differentiating action. Similarly it is seen that

$$\text{if} \quad \mathcal{D}_c Y = a_c, \quad \text{then} \quad Y \quad \text{is} \quad \text{cov.}\, R, \tag{V.a}$$

or the transformer of Y is $Q[\quad]Q_c$. Here the meaning of $\mathcal{D}_c$ is of course, according to (11),

$$\mathcal{D}_c = \frac{\partial}{\partial \ell} - \nabla. \tag{11'}$$

Notice that X and Y may be but are not necessarily full quaternions;[12] they can be, for example, pure vectors, either real (or ordinary vectors) or complex, i.e. *bivectors*, if we are to retain Hamilton's terminology.

Let us now pass to consider the fundamental electromagnetic equations "for the vacuum," as they are recently called, i.e. the system of differential equations

$$\left.\begin{aligned} \frac{\partial \mathbf{E}}{dt} + \rho \mathbf{p} &= c.\,\text{curl}\,\mathbf{M}, \qquad \text{div}\,\mathbf{E} = \rho \\ \frac{\partial \mathbf{M}}{dt} &= -c.\,\text{curl}\,\mathbf{E}, \qquad \text{div}\,\mathbf{M} = 0 \end{aligned}\right\} \tag{18}$$

where $\mathbf{E}$, $\mathbf{M}$ are the electric and magnetic vectors of the field, respectively, ρ the volume-density of electricity and $\mathbf{p}$ the vectorial velocity of its motion, both ρ and $\mathbf{p}$ being given functions of space and time.

First, to condense these equations, put together the electric and the magnetic vectors to make up the *electromagnetic bivector* (or the bivector *of the field*)

$$\mathbf{F} = \mathbf{M} - \imath \mathbf{E} \tag{19}$$

and write again $\ell = \imath ct$. Both *curl* and *div* being distributive, this will give us instead of the four vector equations (18) the two bivectorial equations[13]

$$\frac{\partial \mathbf{F}}{\partial \ell} + \text{curl}\,\mathbf{F} = \frac{1}{c}\rho \mathbf{p}; \qquad \text{div}\,\mathbf{F} = -\imath \rho,$$

[12]This has no influence on their transformational peculiarities as expressed in the above quaternionic form.

[13]The reader will find these equations together with the corresponding bivectorial form of the density of energy and the Poynting flux in my paper published in 1907 in the *Annalen der Physik*, vol. xxii., and (supplement) vol. xxiv. I was then unaware of their possible application to the present purpose. (The η of that paper is the above $\imath\,\mathbf{F}$.)

or, using Hamilton's symbols,

$$\frac{\partial \mathbf{F}}{\partial \ell} + \mathcal{V}\nabla\mathbf{F} = \frac{1}{c}\rho\mathbf{p}; \qquad \mathcal{S}\nabla\mathbf{F} = \iota\rho.$$

Now, remembering that $\mathcal{V}\nabla\mathbf{F} + \mathcal{S}\nabla\mathbf{F} = \nabla\mathbf{F}$ and using the quaternionic differentiator $\mathcal{D}$, explained by (11), the last two coalesce at once into the single equation

$$\mathcal{D}\mathbf{F} = C, \qquad (VI.)$$

in which C is the current-quaternion, as defined by (10).

Thus *the whole system of four equations* (18), the fundamental equations of the electron theory, are represented by *one* quaternionic equation, $(VI.)$.

This condensation is even more complete than in Minkowski's matrix-form, which consists of *two* equations, $\operatorname{lor} f = -s$, $\operatorname{lor} f^* = 0$ (loc. cit. 12), one for the first pair of (18) and the other for the second pair, or in Sommerfeld's equivalent four-dimensional vector form: $\mathbf{Div}\, f = P$ and $\mathbf{Div}\, f^* = 0$ (loc. cit., 5). Here P is the "Viererverktor" corresponding to the current-quaternion C, and f the "Sechservektor," corresponding to the bivector $\mathbf{F}$, while f^* is the "supplement" (Ergänzung) of f, which is another "Sechservektor," though very nearly related to f. Minkowski's f is an alternating matrix of 4×4 elements. But let us return to our quaternionic differential equation $(VI.)$.

C is a (given) physical quaternion cov. q. The operator $\mathcal{D}$ has also the structure of q. What is the relativistic transformer of $\mathbf{F}$? By $(V.)$ we see at once that it is

$$Q_c[\quad]Q,$$

or that $\mathbf{F}$ is transformed like a (scalarless) *L-quaternion*. Thus, the answer is already contained in $(V.)$. But to see clearly the true meaning of the process implied in the relativistic transformation, let us repeat again the whole reasoning somewhat more explicitly. We have, in the system S, as an expression of the laws of electromagnetic phenomena, the equation

$$\mathcal{D}\mathbf{F} = C. \qquad (S)$$

Now, what the Principle of Relativity requires is the same form of the law in the system S', i.e.

$$\mathcal{D}'\mathbf{F}' = C' \qquad (S')$$

Suppose also that both of these equations have been fully confirmed by experience. How are $\mathbf{F}'$ and $\mathbf{F}$ correlated? To adopt language adapted to the general case, use in the accented law or equation (S') the transformer already known, *i.e.* in our present ease $Q[\quad]Q$ for both $\mathcal{D}$ and C; then it becomes

$$Q\,\mathcal{D}Q\mathbf{F}' = QCQ, \quad \text{or} \quad \mathcal{D}Q\mathbf{F}' = CQ,$$

or, by the non-accented equation (S),

$$\mathcal{D}Q\mathbf{F}' = \mathcal{D}\mathbf{F}Q.$$

Hence, rejecting an additive function of obvious properties, *i.e.* requiring that $\mathbf{F}'$ shall vanish together with $\mathbf{F}$,

$$Q\mathbf{F}' = \mathbf{F}Q,$$

or finally, Q being a unit-quaternion,

$$\mathbf{F}' = Q_c\mathbf{F}Q, \qquad (VII.)$$

which is the required correlation, identical with the above.[14] Henceforth we shall have to admit, in the name of Relativity, bivectors transforming like this calling them, say, *physical bivectors* (or in Minkowski's way, "world"-bivectors). Or we can make the *L-quaternion* (of which $\mathbf{F}$ is the vector part) the master, calling it, say, a (*left*) *physical quaternion of the* II. *kind*, and writing $\mathbf{F}$ as its special case

$$\mathbf{F} = \mathcal{V}L = \mathcal{V}a_cb. \qquad (20)$$

(The supplementary scalar, $\mathcal{S}a_cb$, necessary to convert $\mathbf{F}$ into a full quaternion, would present no difficulties, since it has been proved to be an invariant.) The short

[14]Our quaternionic formula $(VII.)$ resembles entirely Minkowski's

$$f' = A^{-1}fA,$$

in which A is a matrix of 4×4 elements, and A^{-1} its reciprocal; *loc. cit.*. §11. The reason of this analogy will easily be seen to depend on the circumstance that both the product of quaternions and the product of matrices have the *associative property*. But at any rate the multiplication by a quaternion, like Q or Q_c, is actually done in a much more simple way than the application of a matrix of 4×4 elements.

Observe also that the above analogy does not extend to the transformation of Minkowski's vectors of the I. kind and our physical quaternions in fact, here the matrix-form is

$$s' = sA, \quad \text{with} \quad s = |s_1, s_2, s_3, s_4|,$$

whereas the quaternionic form is

$$q' = QqQ.$$

name *physical quaternion* might then continue to stand for *physical quaternion of the first kind*, of which q is the standard.

But leave aside questions of nomenclature and return to $(VII.)$. To verify this short formula remember that, by $(I.)$,

$$Q = \sqrt{(1+\gamma)/2} + \mathbf{u}\sqrt{(1-\gamma)/2}, \quad Q_c = \sqrt{(1+\gamma)/2} - \mathbf{u}\sqrt{(1-\gamma)/2},$$

and expand the right side of $(VII.)$. Then

$$\mathbf{F}' = (1-\gamma)(\mathbf{Fu})\mathbf{u} + \gamma\mathbf{F} + \iota\beta\gamma\mathcal{V}\mathbf{Fu}, \tag{21}$$

or splitting into the real and imaginary parts and remembering (19),

$$\left.\begin{aligned} \mathbf{E}' &= (1-\gamma)(\mathbf{Eu})\mathbf{u} + \gamma\mathbf{E} + \beta\gamma\mathcal{V}\mathbf{uM} \\ \mathbf{M}' &= (1-\gamma)(\mathbf{Mu})\mathbf{u} + \gamma\mathbf{M} - \beta\gamma\mathcal{V}\mathbf{uE} \end{aligned}\right\} \tag{21a}$$

Now, these equations give immediately for the components taken along $\mathbf{u}$ (the direction of motion)

$$E_1' = E_1; \qquad M_1' = M_1,$$

and for the two other pairs of rectangular components (the right-handed system being used)

$$E_2' = \gamma(E_2 - \beta M_3); \qquad M_2' = \gamma(M_2 + \beta E_3)$$
$$E_3' = \gamma(E_3 + \beta M_2); \qquad M_3' = \gamma(M_3 - \beta E_2),$$

which are precisely the well-known transformational formulæ, obtained for the first time by Einstein. Thus $(VII.)$ is verified.

Again, Q, Q_c, being unit-quaternions, we see from $(VII.)$ that, as already has been remarked, the tensor of $\mathbf{F}$ is an *invariant*,

$$\mathcal{T}\mathbf{F}' = \mathcal{T}\mathbf{F}, \tag{$VIII.$}$$

which may also be written, more conveniently,[15] $\mathbf{F}'^2 = \mathbf{F}^2$. Now, by (19), $-\mathbf{F}^2 = M^2 - E^2 - 2\iota(\mathbf{EM})$; thus we see that $(VIII.)$ contains *both* of the well-known invariants of Minkowski:

$$M^2 - E^2 \quad \text{and} \quad (\mathbf{EM}) \tag{22}$$

Notice that what is called a *pure* electromagnetic wave is defined by $M^2 = E^2$, $(\mathbf{EM}) = 0$. Using the above form we can characterize a pure wave more simply by[16]

$$\mathcal{T}\mathbf{F} = 0, \quad \text{or} \quad \mathbf{F}^2 = \mathbf{FF} = 0.$$

[15] Remember that, $\mathbf{F}$ being a *scalarless* quaternion, its conjugate is simply $-\mathbf{F}$.

[16] This remark will be found also in my paper of 1907, cited above.

Thus, by $(VIII.)$, a wave which is pure to the S-inhabitants, is also pure to the S'-inhabitants. But this example only by the way.

Instead of the above $\mathbf{F}$, as defined by (19), we may as well take the *complementary bivector*

$$\mathbf{G} = \mathbf{M} + \imath\mathbf{E}.^{17} \tag{19a}$$

Then we shall get as the quaternionic equivalent of the electromagnetic equations (18), instead of and in quite the same way as $(VI.)$,

$$\mathcal{D}_c\,\mathbf{G} = C_c, \tag{VI.a}$$

where C_c is the conjugate current-quaternion $\rho(\imath - \mathbf{p}/c)$ and $\mathcal{D}_c$ the conjugate differential operator $\partial/\partial\ell - \nabla$, as already explained.

We now see, by $(V.a)$, that $\mathbf{G}$ is transformed like an *R-quaternion*, i.e.

$$\mathbf{G}' = Q\mathbf{G}Q_c. \tag{VII.a}$$

Again we may write, similarly to (20),

$$\mathbf{G} = \mathcal{V}R = \mathcal{V}de_c, \tag{20a}$$

d, e_c being a pair of physical quaternions covariant with q and q_c respectively. And since $\mathbf{G}$ is a *physical bivector*, just as much as $\mathbf{F}$, we may again call $R = de_c$ a (*right*) *physical quaternion of the second kind*.

Notice that, at least for the time being, we have no need of both $\mathbf{F}$ and $\mathbf{G}$, since we require either $\mathbf{F}$ only or $\mathbf{G}$ only. (Possibly for the further development of Quaternionic Relativity the simultaneous use of $\mathbf{F}$, $\mathbf{G}$ may turn out to be convenient or even necessary.)

As regards the relation of $(20a)$ to (20), observe that generally we cannot write $d = a$, $e = b$; in fact, the reader will easily prove for himself that this would require $(\mathbf{EM}) = 0$, i.e. $\mathbf{E}\perp\mathbf{M}$, and would not, consequently, be sufficiently general. The only essential thing here is that in (20) it is the first and in $(20a)$ the second factor which has the subscript c. This is shown also by the symbols L (left), R (right).

Let us return to the quaternionic differential equation for the vacuum, in its first form, i.e.

$$\mathcal{D}\mathbf{F} = C. \tag{VI.}$$

[17] $\mathbf{G}$ is a complex vector "conjugate" to $\mathbf{F}$, in the sense of the word used in the Theory of Functions. But to avoid confusion with the quaternionic notion of conjugate, I do not call it by this name and do not denote it by $\mathbf{F}_c$.

Remember that $\mathcal{D}\,\mathcal{D}_c = (\mathcal{T}\,\mathcal{D})^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \ell^2}$ is the four-dimensional Laplacian, or Cauchy's $\Box$,

$$\mathcal{D}\,\mathcal{D}_c = \Box \qquad (IX.)$$

Hence, if $\mathbf{\Phi}$ be an auxiliary quaternion and if we put $\mathbf{F} = -\,\mathcal{V}\,\mathcal{D}_c\,\mathbf{\Phi}$ (since $\mathbf{F}$ is *scalarless*), or more simply if we write

$$\mathbf{F} = -\,\mathcal{D}_c\,\mathbf{\Phi} \qquad (X.)$$

demanding at the same time that

$$\mathcal{S}\,\mathcal{D}_c\,\mathbf{\Phi} = 0, \qquad (XI.)$$

then we get at once from $(VI.)$

$$\Box\mathbf{\Phi} = -C, \qquad (XII.)$$

which is the well-known equation, obtained by Sommerfeld for his "Viererpotential." But here, I daresay, it follows from $(VI.)$ more immediately, than by the use of four-dimensional divergences and curls or "Rotations."

The above $\mathbf{\Phi}$, which may be called the *potential-quaternion*, is easily proved to be a *physical quaternion*, namely, cov. q. For by its definition, $(X.)$, and remembering that $\mathbf{F}$ is cov. L, we have immediately

$$\mathbf{\Phi}\,\text{cov.}\,\mathcal{D}_c^{-1}\,\mathbf{F}\,\text{cov.}\,\mathcal{D}\,\mathbf{F}\,\text{cov.}\,\mathcal{D}L,$$

i.e., by $(IV.)$, $\mathbf{\Phi}$ cov. q,—q.e.d.[18]

Writing the potential-quaternion

$$\mathbf{\Phi} = \imath\phi + \mathbf{A}, \qquad (23)$$

where ϕ is a real scalar and $\mathbf{A}$ a real vector, it is seen at once that ϕ is the ordinary "scalar potential" and $\mathbf{A}$ the ordinary "vector potential." In fact, developing $(X.)$ we have

$$\mathbf{F} = \mathcal{V}\nabla\mathbf{A} - \frac{\partial\mathbf{A}}{\partial\ell} + \imath\nabla\phi = \mathbf{M} - \imath\mathbf{E},$$

whence the usual formulæ

$$\mathbf{M} = \mathcal{V}\nabla\mathbf{A} = \operatorname{curl}\mathbf{A},$$

[18] This is seen even more immediately from $(XII.)$. For, since $\Box = (\mathcal{T}\,\mathcal{D})^2$ is an invariant, $\mathbf{\Phi}$ is transformed like C and, consequently, like q.

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial A}{\partial t}.$$

Also the condition $(XI.)$ is expanded immediately into the usual equation

$$\frac{1}{c}\frac{\partial\phi}{\partial t} + \operatorname{div}\mathbf{A} = 0.$$

Finally, notice that the "equation of continuity," as it is commonly called, *i.e.*

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{p}) = 0,$$

assumes the quaternionic form

$$\mathcal{S}\,\mathcal{D}_c C = 0. \qquad (XIII.)$$

The scalar of $\mathcal{D}_c$ is, in fact, the same thing as Sommerfeld's four-dimensional divergence **Div**.

Or we may write, equivalently,

$$\mathcal{S}\,\mathcal{D} C_c = 0. \qquad (XIIIa.)$$

We know already that the electromagnetic bivector **F** is a (scalarless) L-quaternion. Hence, by $(IV.)$, if we multiply it, on the left side, by any physical quaternion cov. q, the resulting product will again be transformed like q. Now, the current-quaternion C being precisely such a quaternion, consider the product

$$P = C\mathbf{F}, \qquad (24)$$

which, by the above, will again be transformed by $Q[\quad]Q$. Develop it, by (10) and (19); then

$$P = \rho\left\{\imath\mathbf{M} + \mathbf{E} + \frac{1}{c}\mathbf{pM} - \frac{\imath}{c}\mathbf{pE}\right\}$$

or, remembering that the full product AB is $\mathcal{V}AB - (AB)$,

$$P = P_e + \imath P_m, \qquad (25)$$

where P_e, P_m are the quaternions

$$P_e = \rho\left\{\frac{\imath}{c}(\mathbf{pE}) + \mathbf{E} + \frac{1}{c}\,\mathcal{V}\mathbf{pM}\right\} \qquad (25e)$$

$$P_m = \rho\left\{\frac{\imath}{c}(\mathbf{pM}) + \mathbf{M} - \frac{1}{c}\,\mathcal{V}\mathbf{pE}\right\} \qquad (25m)$$

The vector of P_e is the well-known *ponderomotive force*, per unit volume, and the scalar of P_e is ι/c times the *activity* of this force, while P_m is the magnetic analogue of P_e. Notice that the whole P, (25), though having with q the transformer $Q[\quad]Q$ in common, has *not* the structure of the standard q, inasmuch as it is a full *biquaternion*.[19] (And how each of its constituents, P_e, P_m, which *have* the structure of q, are transformed, we do not as yet know,—though we shall know in a moment.)

Similarly, the complementary electromagnetic bivector **G** being a (scalarless) R-quaternion, multiply it on the *right* side by C. Then the product $\mathbf{G}C$ will, by $(IV.)$, again be transformed by $Q[\quad]Q$, *i.e.*, again like q. Develop it; then, by (10) and (19a),

$$\mathbf{G}C = \rho\left\{\iota\mathbf{M} + \frac{1}{c}\mathbf{Mp} - \mathbf{E} + \frac{\iota}{c}\mathbf{Ep}\right\},$$

and this is precisely, with the same meanings of P_e and P_m as above, equal to

$$\mathbf{G}C = -P_e + \iota P_m. \tag{26}$$

This again is a full biquaternion.

Now, both biquaternions, $P = C\mathbf{F}$ and $\mathbf{G}C$ being transformed by the same $Q[\quad]Q$, this will also be the transformer of their sum, and of their difference, *i.e.*, by (25) and (26), of P_m and of P_e.

Thus we see that not only P but also its constituents P_e and P_m, taken separately, are cov. q; and since each of them has also the structure of q,[20] *both P_e and P_m are physical quaternions*, cov. q.

They are given explicitly by (25e), (25m), and may, by the above, be written also

$$P_e = \frac{1}{2}\{C\mathbf{F} - \mathbf{G}C\} \tag{27e}$$

$$P_m = -\frac{\iota}{2}\{C\mathbf{F} + \mathbf{G}C\} \tag{27m}$$

It is true that (at least on the ground of the fundamental electronic equations) only P_e has an immediate physical meaning, and not P_m. But this does not seem to me a disadvantage. On the contrary; since our stock of physical quaternions, as the reader will certainly have observed, is as yet not very big, it may be better to have one more.

P_e corresponds to the "Viererkraft"[21] and might consequently be called here the *force-quaternion*. It has a dynamic vector and an energetic scalar, as observed

[19] In *Hamilton's*, of course, and not in *Clifford's* meaning of the word.
[20] Namely an imaginary scalar and a real vector.
[21] See Laue, *loc. cit.*, 15.

above. As to P_m, it is of no importance to give it (at least for the "vacuum") any special name. On the other hand, the *whole* P, which may possibly turn out to be more convenient for the quaternionic treatment of Relativity, might be called the *dynamical*[22] *biquaternion*, and be looked on as the standard of *physical biquaternions*, in the same manner as q, **F** have been the standards of physical quaternions and of physical bivectors, respectively.[23]

Now, using the quaternionic differential equation $(VI.)$, or $C = \mathcal{D}\mathbf{F}$, the formula $(27e)$ for P_e may be written

$$2P_e = \mathcal{D}\mathbf{F}.\mathbf{F} - \mathbf{G}.\mathcal{D}\mathbf{F}, \tag{28}$$

and similarly $(27m)$ for P_m, the *dot* being a separator, as regards the differentiating action of $\mathcal{D}$. In (28) the force-quaternion P_e, is immediately expressed by the electromagnetic bivector **F** and its complementary **G**. Thus, the formula (28) is adapted for showing the properties of the Maxwellian stress and of the electromagnetic momentum along with the flux and the density of energy, in correspondence to the equivalent formula of Minkowski's four-dimensional system.

But, since we already know everything about the behaviour of each constituent of P, *i.e.* of P_e, P_m, we may dismiss them altogether and use more conveniently the full *dynamical biquaternion* P, as defined by (24). Thus, using again the equation $(VI.)$, we shall have, more simply,

$$P = \mathcal{D}[\mathbf{F}.\mathbf{F}], \tag{XIV.}$$

where the purpose of the brackets is only to emphasize the circumstance that **F**.**F** plays the part of a *dyad*. This will lead us to the quaternionic treatment of questions regarding stress, and localization and flux of energy

But these fundamental dynamical questions will best be postponed and reserved for a future publication, in which also the quaternionic treatment of the electrodynamics of ponderable bodies and of some other relativistic subjects will be given.

November, 1911.

XVIII. Intelligence and Miscellaneous Articles.

[22]Notwithstanding that it is partially energetic.

[23]It is worth noticing again that **F** (*plus* an invariant and consequently unessential scalar) and P may be regarded as alternating products of 2 and of 3 physical quaternions, respectively. From this standpoint q, **F**, P and their respective companions might he considered as quaternionic entities of the 1st, 2nd, and 3rd *degree*, respectively.

Phil. Mag. 24 (1912), 208.

THE QUATERNIONIC FORM OF RELATIVITY

To the Editors of the Philosophical Magazine.

GENTLEMEN,—

The appearance of Prof. Silberstein's paper entitled "The Quaternionic Form of Relativity" in the May issue of the Phil. Mag. is a welcome sign that continental mathematicians, who have already largely availed themselves of various systems of vector notation, are perhaps awakening to the suitability of quaternions in such a connexion. Minkowsky (see footnote p. 79, *Nach. Gött.* 1908) must have had such an idea, but decided in favour of the matrix notation. The quaternion form of the Hertz Heaviside equations was given by the present writer at the British Association Meeting at Dublin (1908), and had been given by him in lectures for some years previously. An application of quaternions to the Relativity Principle will be found in a paper, vol. xxix. Section A, No. 1, Proc. Irish Academy (read Feb. 1911). Other writers, such as Somerfeld, subsequent to Minkowsky, have used the four-dimensional vector. The obvious defect of this latter method is that it involves a second kind of vector having six components. Beyond this, however, the quaternion has the advantage of being asymmetrical, the time-scalar occupying a different position from the space-vector. It is thus more in touch with real phenomena. For no matter what view we take of relativity, the physical methods of measuring time are quite different from those of measuring spaces, and the flexibility of the quaternion product allows us to put our results at once in a form suitable, if need be, for numerical computation.

Yours truly,

ARTHUR W. CONWAY