Quaternionic analysis

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1. *Introduction*. The richness of the theory of functions over the complex field makes it natural to look for a similar theory for the only other non-trivial real associative division algebra, namely the quaternions. Such a theory exists and is quite far-reaching, yet it seems to be little known. It was not developed until nearly a century after Hamilton's discovery of quaternions. Hamilton himself (**1**) and his principal followers and expositors, Tait (**2**) and Joly (**3**), only developed the theory of functions of a quaternion variable as far as it could be taken by the general methods of the theory of functions of several real variables (the basic ideas of which appeared in their modern form for the first time in Hamilton's work on quaternions). They did not delimit a special class of regular functions among quaternion-valued functions of a quaternion variable, analogous to the regular functions of a complex variable.

This may have been because neither of the two fundamental definitions of a regular function of a complex variable has interesting consequences when adapted to quaternions; one is too restrictive, the other not restrictive enough. The functions of a quaternion variable which have quaternionic derivatives, in the obvious sense, are just the constant and linear functions (and not all of them); the functions which can be represented by quaternionic power series are just those which can be represented by power series in four real variables.

In 1935, R. Fueter (**4**) proposed a definition of 'regular' for quaternionic functions by means of an analogue of the Cauchy-Riemann equations. He showed that this definition led to close analogues of Cauchy's theorem, Cauchy's integral formula, and the Laurent expansion (**5**). In the next twelve years Fueter and his collaborators developed the theory of quaternionic analysis. A complete bibliography of this work is contained in (**6**), and a simple account (in English) of the elementary parts of the theory has been given by Deavours (**7**).

The theory developed by Fueter and his school is incomplete in some ways, and many of their theorems are neither so general nor so rigorously proved as present-day standards of exposition in complex analysis would require. The purpose of this paper is to give a self-contained account of the main line of quaternionic analysis which remedies these deficiencies, as well as adding a certain number of new results. By using the exterior differential calculus we are able to give new and simple proofs of most of the main theorems and to clarify the relationship between quaternionic analysis and complex analysis.

In Section 2 of this paper we establish our notation for quaternions, and introduce the quaternionic differential forms dq, $dq \wedge dq$ and Dq, which play a fundamental role in quaternionic analysis. The 1-form dq and the 3-form Dq have simple geometrical interpretations as the tangent to a curve and the normal to a hypersurface, respectively.

Section 3 is concerned with the definition of a regular function. The remarks in the second paragraph of this introduction, about possible analogues of the definition of a complex-analytic function, are amplified (this material seems to be widely known, but is not easily accessible in the literature); then Fueter's definition of a regular function, by means of an analogue of the Cauchy-Riemann equations, is shown to be equivalent to the existence of a certain kind of quaternionic derivative. Just as, for a function $f : \mathbb{C} \to \mathbb{C}$, the Cauchy-Riemann equation $\partial f/\partial x + i\,\partial f/\partial y = 0$ (the variable being $z = x + iy$) is equivalent to the existence of a complex number $f'(z)$ such that $df = f'(z)\,dz$, so for a function $f : \mathbb{H} \to \mathbb{H}$, the Cauchy-Riemann-Fueter equation

$$\frac{\partial f}{\partial t} + i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z} = 0 \tag{1.1}$$

(the variable being $q = t + ix + jy + kz$) is equivalent to the existence of a quaternion $f'(q)$ such that $d(dq \wedge dqf) = Dq\,f'(q)$.

Section 4 is devoted to the quaternionic versions of Cauchy's theorem and Cauchy's integral formula. If the function f is continuously differentiable and satisfies (1.1), Gauss's theorem can be used to show that

$$\int_{\partial C} Dq\,f = 0, \tag{1.2}$$

where C is any smooth closed 3-manifold in $\mathbb{H}$, and that if q_0 lies inside C,

$$f(q_0) = \frac{1}{2\pi^2}\int_{\partial C} \frac{(q-q_0)^{-1}}{|q-q_0|^2}\,Dq\,f(q). \tag{1.3}$$

We will show that Goursat's method can be used to weaken the conditions on the contour C and the function f, so that C need only be assumed to be rectifiable and the derivatives of f need not be continuous. From the integral formula (1.3) it follows, as in complex analysis, that if f is regular in an open set U then it has a power series expansion about each point of U. Thus pointwise differentiability, together with the four real conditions (1.1) on the 16 partial derivatives of f, are sufficient to ensure real-analyticity.

In Section 5 we show how regular functions can be constructed from functions of more familiar type, namely harmonic functions of four real variables and analytic functions of a complex variable, and how a regular function gives rise to others by conformal transformation of the variable.

The homogeneous components in the power series representing a regular function are themselves regular; thus it is important to study regular homogeneous polynomials, the basic functions from which all regular functions are constructed. The corresponding functions of a complex variable are just the powers of the variable, but the situation with quaternions is more complicated. The set of homogeneous regular functions of degree n forms a quaternionic vector space of dimension $\frac{1}{2}(n+1)(n+2)$; this is true for any integer n if for negative n it is understood that the functions are defined and regular everywhere except at 0. The functions with negative degree of homogeneity correspond to negative powers of a complex variable and occur in the quaternionic Laurent series which exists for any function which is regular in an open set except at one point. Fueter found two natural bases for the set of homogeneous functions which play dual roles in the calculus of residues. (He actually only proved that these bases form spanning sets.) In Section 6 we study homogeneous regular functions by means of harmonic analysis on the unit sphere in $\mathbb{H}$, which forms a group isomorphic to $SU(2)$; this bears the same relation to quaternionic analysis as the theory of Fourier series does to complex analysis. In Section 7 we examine the power series representing a regular function and obtain analogues of Laurent's theorem and the residue theorem.

Many of the algebraic and geometric properties of complex analytic functions are not present in quaternionic analysis. Because quaternions do not commute, regular functions of a quaternion variable cannot be multiplied or composed to give further regular functions. Because the quaternions are four-dimensional, there is no counterpart to the geometrical description of complex analytic functions as conformal mappings. The zeros of a quaternionic regular function are not necessarily isolated, and its range is not necessarily open; neither of these sets need even be a submanifold of $\mathbb{H}$. There is a corresponding complexity in the structure of the singularities of a quaternionic regular function; this was described by Fueter (**9**), but without giving precise statements or proofs. This topic is not investigated here.

2. *Preliminaries*. We denote the four-dimensional real associative algebra of the quaternions by $\mathbb{H}$, its identity by 1, and we regard $\mathbb{R}$ as being embedded in $\mathbb{H}$ by identifying $t \in \mathbb{R}$ with $1 \in \mathbb{H}$. Then we have a vector space direct sum $\mathbb{H} = \mathbb{R} \oplus P$, where P is an oriented three-dimensional Euclidean vector space, and with the usual notation for three-dimensional vectors the product of two elements of P is given by

$$\mathbf{ab} = -\mathbf{a}.\mathbf{b} + \mathbf{a} \times \mathbf{b}. \tag{2.1}$$

We choose an orthonormal positively oriented basis $\{i, j, k\}$ for P, and write a typical quaternion as

$$q = t + ix + jy + kz \qquad (t, x, y, z \in \mathbb{R}). \tag{2.2}$$

On occasion we will denote the basic quaternions i, j, k by e_i and the coordinates x, y, z by $x_i \quad (i = 1, 2, 3)$ and use the summation convention for repeated indices. Then (2.2) becomes

$$q = t + e_i x_i \tag{2.3}$$

and the multiplication is given by

$$e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k. \tag{2.4}$$

We will also sometimes identify the subfield spanned by 1 and i with the complex field $\mathbb{C}$, and write

$$q = v + jw \qquad (v, w \in \mathbb{C}) \tag{2.5}$$

where $v = t + ix$ and $w = y - iz$. The multiplication law is then

$$vj = j\bar{v} \tag{2.6}$$

for all $v \in \mathbb{C}$.

We write

$$\bar{q} = t - ix - jy - kz, \tag{2.7}$$

$$|q| = \sqrt{q\bar{q}} = \sqrt{t^2 + x^2 + y^2 + z^2} \in \mathbb{R}, \tag{2.8}$$

$$\operatorname{Re} q = \tfrac{1}{2}(q + \bar{q}) = t \in \mathbb{R}, \tag{2.9}$$

$$\operatorname{Pu} q = \tfrac{1}{2}(q - \bar{q}) = ix + jy + kz \in P, \tag{2.10}$$

$$\operatorname{Un} q = \frac{q}{|q|} \in S, \tag{2.11}$$

where S is the unit sphere in $\mathbb{H}$; and

$$\langle q_1, q_2 \rangle = \operatorname{Re}(q_1 \bar{q}_2) = t_1 t_2 + x_1 x_2 + y_1 y_2 + z_1 z_2. \tag{2.12}$$

Then we have

$$\overline{q_1 q_2} = \bar{q}_2 \, \bar{q}_1, \tag{2.13}$$

$$|q_1 q_2| = |q_1| \cdot |q_2|, \tag{2.14}$$

$$\operatorname{Re}(q_1 q_2) = \operatorname{Re}(q_2 q_1) \tag{2.15}$$

and

$$q^{-1} = \frac{\bar{q}}{|q|^2}. \tag{2.16}$$

Note that if u_1 and u_2 are unit quaternions, i.e. $|u_1| = |u_2| = 1$, the map $q \mapsto u_1 q u_2$ is orthogonal with respect to the inner product (2.12) and has determinant 1; conversely, any rotation of $\mathbb{H}$ is of the form $q \mapsto u_1 q u_2$ for some $u_1, u_2 \in \mathbb{H}$ (see, for example, (**10**), chap. 10).

The inner product (2.12) induces an $\mathbb{R}$-linear map $\Gamma : \mathbb{H}^* \to \mathbb{H}$, where

$$\mathbb{H}^* = \operatorname{Hom}_{\mathbb{R}}(\mathbb{H}, \mathbb{R})$$

is the dual vector space to $\mathbb{H}$, given by

$$\langle \Gamma(\alpha), q \rangle = \alpha(q) \tag{2.17}$$

for $\alpha \in \mathbb{H}^*$, $q \in \mathbb{H}$. Since $\{1, i, j, k\}$ is an orthonormal basis for $\mathbb{H}$, we have

$$\Gamma(\alpha) = \alpha(1) + i\,\alpha(i) + j\,\alpha(j) + k\,\alpha(k).$$

The set of $\mathbb{R}$-linear maps from $\mathbb{H}$ to $\mathbb{H}$ forms a two-sided vector space over $\mathbb{H}$ of dimension 4, which we will denote by F_1. It is spanned (over $\mathbb{H}$) by $\mathbb{H}^*$, so the map Γ can be extended by linearity to a right $\mathbb{H}$-linear map $\Gamma_r : F_1 \to \mathbb{H}$ and a left-linear map

$$\Gamma_l : F_1 \to \mathbb{H}.$$

They are given by

$$\Gamma_r(\alpha) = \alpha(1) + i\,\alpha(i) + j\,\alpha(j) + k\,\alpha(k) \tag{2.18}$$

and

$$\Gamma_l(\alpha) = \alpha(1) + \alpha(i)\, i + \alpha(j)\, j + \alpha(k)\, k \tag{2.19}$$

for any $\alpha \in F_1$.

The geometric terminology used in this paper is as follows:

An *oriented k-parallelepiped* in $\mathbb{H}$ is a map $C : I^k \to \mathbb{H}$, where $I^k \subset \mathbb{R}^k$ is the closed unit k-cube, of the form

$$C(t_1, ..., t_k) = q_0 + t_1 h_1 + ... + t_k h_k.$$

$q_0 \in \mathbb{H}$ is called the *original vertex* of the parallelepiped, and $h_1, ..., h_k \in \mathbb{H}$ are called its *edge-vectors*. A parallelepiped is *non-degenerate* if its edge-vectors are linearly independent (over $\mathbb{R}$). A non-degenerate 4-parallelepiped is *positively oriented* if

$$v(h_1, ..., h_4) > 0,$$

negatively oriented if $v(h_1, ..., h_4) < 0$, where v is the volume form defined below (equation (2.26)).

We will sometimes abuse notation by referring to the image $C(I^k)$ as simply C.

Quaternionic differential forms. When it is necessary to avoid confusion with other senses of differentiability which we will consider, we will say that a function $f : \mathbb{H} \to \mathbb{H}$ is *real-differentiable* if it is differentiable in the usual sense. Its differential at a point $q \in \mathbb{H}$ is then an $\mathbb{R}$-linear map $df_q : \mathbb{H} \to \mathbb{H}$. By identifying the tangent space at each point of $\mathbb{H}$ with $\mathbb{H}$ itself, we can regard the differential as a quaternion-valued 1-form

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dz. \tag{2.20}$$

Conversely, any quaternion-valued 1-form $\theta = a_0 dt + a_i dx_i (a_0, a_i \in \mathbb{H})$ can be regarded as the $\mathbb{R}$-linear map $\theta : \mathbb{H} \to \mathbb{H}$ given by

$$\theta(t + x_i e_i) = a_0 t + a_i x_i \tag{2.21}$$

Similarly, a quaternion-valued r-form can be regarded as a mapping from $\mathbb{H}$ to the space of alternating $\mathbb{R}$-multilinear maps from $\mathbb{H} \times ... \times \mathbb{H}$ (r times) to $\mathbb{H}$. We define the exterior product of such forms in the usual way: if θ is an r-form and ϕ is an s-form,

$$\theta \wedge \phi(h_1, ..., h_{r+s}) = \frac{1}{r!s!} \sum_{\rho} \epsilon(\rho) \theta(h_{\rho(1)}, ..., h_{\rho(r)}) \phi(h_{\rho(r+1)}, ..., h_{\rho(r+s)}), \tag{2.22}$$

where the sum is over all permutations ρ of $r + s$ objects, and $\epsilon(\rho)$ is the sign of ρ. Then the set of all r-forms is a two-sided quaternionic vector space, and we have

$$\left.\begin{aligned} a(\theta \wedge \phi) &= (a\theta) \wedge \phi, \\ (\theta \wedge \phi)a &= \theta \wedge (\phi a), \\ (\theta a) \wedge \phi &= \theta \wedge (a\phi) \end{aligned}\right\} \tag{2.23}$$

for all quaternions a, r-forms θ and s-forms ϕ. The space of quaternionic r-forms has a basis of real r-forms, consisting of exterior products of the real 1-forms dt, dx,

dy, dz; for such forms left and right multiplication by quaternions coincide. Note that because the exterior product is defined in terms of quaternion multiplication, which is not commutative, it is not in general true that $\theta \wedge \phi = -\phi \wedge \theta$ for quaternionic 1-forms θ and ϕ.

The exterior derivative of a quaternionic differential form is defined by the usual recursive formulae, and Stokes's theorem holds in the usual form for quaternionic integrals.

The following special differential forms will be much used in the rest of the paper. The differential of the identity function is

$$dq = dt + i\,dx + j\,dy + k\,dz; \tag{2.24}$$

regarded as an $\mathbb{R}$-linear transformation of $\mathbb{H}$, dq is the identity mapping. Its exterior product with itself is

$$dq \wedge dq = \tfrac{1}{2}\epsilon_{ijk}\, e_i\, dx_j \wedge dx_k = i\,dy \wedge dz + j\,dz \wedge dx + k\,dx \wedge dy, \tag{2.25}$$

which, as an antisymmetric function on $\mathbb{H} \times \mathbb{H}$, gives the commutator of its arguments. For the (essentially unique) constant real 4-form we use the abbreviation

$$v = dt \wedge dx \wedge dy \wedge dz, \tag{2.26}$$

so that $v(1, i, j, k) = 1$. Finally, the 3-form Dq is defined as an alternating $\mathbb{R}$-trilinear function by

$$\langle h_1, Dq(h_2, h_3, h_4)\rangle = v(h_1, h_2, h_3, h_4) \tag{2.27}$$

for all $h_1, ..., h_4 \in \mathbb{H}$. Thus $Dq(i, j, k) = 1$ and $Dq(1, e_i, e_j) = -\epsilon_{ijk}e_k$. The coordinate expression for Dq is

$$\begin{aligned} Dq &= dx \wedge dy \wedge dz - \tfrac{1}{2}\epsilon_{ijk}\, e_i\, dt \wedge dx_j \wedge dx_k \\ &= dx \wedge dy \wedge dz - i\,dt \wedge dy \wedge dz - j\,dt \wedge dz \wedge dx - k\,dt \wedge dx \wedge dy. \end{aligned} \tag{2.28}$$

Geometrically, $Dq(a, b, c)$ is a quaternion which is perpendicular to a, b and c and has magnitude equal to the volume of the 3-dimensional parallelepiped whose edges are a, b and c. It also has the following algebraic expression:

PROPOSITION 1. $Dq(a, b, c) = \frac{1}{2}(c\bar{a}b - b\bar{a}c)$.

Proof. For any unit quaternion u, the map $q \mapsto uq$ is an orthogonal transformation of $\mathbb{H}$ with determinant 1; hence

$$Dq(ua, ub, uc) = u\, Dq(a, b, c).$$

Taking $u = |a|a^{-1}$, and using the $\mathbb{R}$-trilinearity of Dq, we obtain

$$Dq(a, b, c) = |a|^2 a\, Dq(1, a^{-1}b, a^{-1}c). \tag{2.29}$$

Now since $Dq(1, e_i, e_j) = -\epsilon_{ijk}e_k = \frac{1}{2}(e_je_i - e_ie_j)$, we have by linearity

$$Dq(1, h_1, h_2) = \tfrac{1}{2}(h_2h_1 - h_1h_2) \tag{2.30}$$

for all $h_1, h_2 \in \mathbb{H}$. Hence

$$\begin{aligned} Dq(a,b,c) &= \tfrac{1}{2}|a|^2 a(a^{-1}ca^{-1}b - a^{-1}ba^{-1}c) \\ &= \tfrac{1}{2}(c\bar{a}b - b\bar{a}c). \qquad \square \end{aligned}$$

Two useful formulae were obtained in the course of this proof. The argument leading to (2.29) can be generalized, using the fact that the map $q \mapsto uqv$ is a rotation for any pair of unit quaternions u, v, to

$$Dq(ah_1b, ah_2b, ah_3b) = |a|^2|b|^2 a\, Dq(h_1, h_2, h_3)b; \tag{2.31}$$

and the formula (2.30) can be written as

$$1 \rfloor Dq = -\tfrac{1}{2}\, dq \wedge dq, \tag{2.32}$$

where $\rfloor$ denotes the usual inner product between differential forms and vector fields and 1 denotes the constant vector field whose value is 1.

Since the differential of a quaternion-valued function on H is an element of F_1, the map Γ_r can be applied to it. The result is

$$\Gamma_r(df) = \frac{\partial f}{\partial t} + i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z}. \tag{2.33}$$

We introduce the following notation for the differential operator occurring in (2.33), and for other related differential operators:

$$\left.\begin{aligned} \bar{\partial}_l f = \tfrac{1}{2}\Gamma_r(df) &= \frac{1}{2}\left(\frac{\partial f}{\partial t} + e_i\frac{\partial f}{\partial x_i}\right), \\ \partial_l f &= \frac{1}{2}\left(\frac{\partial f}{\partial t} - e_i\frac{\partial f}{\partial x_i}\right), \\ \bar{\partial}_r f = \tfrac{1}{2}\Gamma_l(df) &= \frac{1}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i}e_i\right), \\ \partial_r f &= \frac{1}{2}\left(\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x_i}e_i\right), \\ \Delta f &= \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \end{aligned}\right\} \tag{2.34}$$

Note that ∂_l, $\bar{\partial}_l$, ∂_r and $\bar{\partial}_r$ all commute, and that

$$\Delta = 4\partial_r\bar{\partial}_r = 4\partial_l\bar{\partial}_l. \tag{2.35}$$

3. *Regular functions*. The requirement that a function of a complex variable $z = x + iy$ should be a complex polynomial, i.e. a sum of terms $a_n z^n$, picks out a proper subset of the polynomial functions $f(x,y) + ig(x,y)$. The corresponding requirement of a function of a quaternion variable $q = t + ix + jy + kz$, namely that it should be a sum of monomials $a_0qa_1...a_{r-1}qa_r$, places no restriction on the function; for in

contrast to the complex case the coordinates $t,\ x,\ y,\ z$ can themselves be written as quaternionic polynomials:

$$\left.\begin{aligned}
t &= \frac{1}{4}(q - iqi - jqj - kqk),\\
x &= \frac{1}{4i}(q - iqi + jqj + kqk),\\
y &= \frac{1}{4j}(q + iqi - jqj + kqk),\\
z &= \frac{1}{4k}(q + iqi + jqj - kqk),
\end{aligned}\right\} \tag{3.1}$$

and so every real polynomial in $t,\ x,\ y,\ z$ is a quaternionic polynomial in q. Thus a theory of quaternionic power series will be the same as a theory of real-analytic functions on $\mathbb{R}^4$.

On the other hand, the requirement that a function of a quaternion variable should have a quaternionic derivative, in the obvious sense, is too strong to have interesting consequences, as we will now show.

Definition. A function $f : \mathbb{H} \to \mathbb{H}$ is *quaternion-differentiable on the left* at q if the limit

$$\frac{df}{dq} = \lim_{h \to 0} [h^{-1}\{f(q+h) - f(q)\}]$$

exists.

THEOREM 1. *Suppose the function f is defined and quaternion-differentiable on the left throughout a connected open set U. Then on U, f has the form*

$$f(q) = a + qb$$

for some $a, b \in \mathbb{H}$.

Proof. From the definition it follows that if f is quaternion-differentiable on the left at q, it is real-differentiable at q and its differential is the linear map of multiplication on the right by $\partial f/\partial q$:

$$df_q(h) = h\,\frac{df}{dq},$$

i.e.

$$df_q = dq\,\frac{df}{dq}. \tag{3.2}$$

Equating coefficients of dt, dx, dy and dz gives

$$\frac{df}{dq} = \frac{\partial f}{\partial t} = -i\,\frac{\partial f}{\partial x} = -j\,\frac{\partial f}{\partial y} = -k\,\frac{\partial f}{\partial z}. \tag{3.3}$$

Put $q = v + jw$, where $v = t + ix$ and $w = y - iz$, and let $f(q) = g(v, w) + jh(v, w)$, where g and h are complex-valued functions of the two complex variables v and w; then (3.3) can be separated into the two sets of complex equations

$$\frac{\partial g}{\partial t} = -i\,\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} = i\,\frac{\partial h}{\partial z},$$

$$\frac{\partial h}{\partial t} = i\,\frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y} = i\,\frac{\partial g}{\partial z}.$$

In terms of complex derivatives, these can be written as

$$\frac{\partial g}{\partial \bar{v}} = \frac{\partial h}{\partial \bar{w}} = \frac{\partial h}{\partial v} = \frac{\partial g}{\partial w} = 0, \tag{3.4}$$

$$\frac{\partial g}{\partial v} = \frac{\partial h}{\partial w} \tag{3.5}$$

and

$$\frac{\partial h}{\partial \bar{v}} = -\frac{\partial g}{\partial \bar{w}}. \tag{3.6}$$

Equation (3.4) shows that g is a complex-analytic function of v and $\bar{w}$, and h is a complex-analytic function of $\bar{v}$ and w. Hence by Hartogs's theorem ((**11**), p. 133) g and h have continuous partial derivatives of all orders and so from (3.5)

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial h}{\partial w}\right) = \frac{\partial}{\partial w}\left(\frac{\partial h}{\partial v}\right) = 0.$$

Suppose for the moment that U is convex. Then we can deduce that g is linear in $\bar{w}$, h is linear in w and h is linear in $\bar{v}$. Thus

$$\begin{aligned} g(v,w) &= \alpha + \beta v + \gamma\bar{w} + \delta v\bar{w}, \\ h(v,w) &= \epsilon + \zeta\bar{v} + \eta w + \theta\bar{v}w, \end{aligned}$$

where the Greek letters represent complex constants. Now (3.5) and (3.6) give the following relations among these constants:

$$\beta = \eta, \quad \zeta = -\gamma, \quad \delta = \theta = 0.$$

Thus

$$\begin{aligned} f &= g + jh = \alpha + j\epsilon + (v + jw)(\beta - j\gamma) \\ &= a + qb, \end{aligned}$$

where $a = \alpha + j\epsilon$ and $b = \beta - j\gamma$; so f is of the stated form if U is convex. The general connected open set can be covered by convex sets, any two of which can be connected by a chain of convex sets which overlap in pairs; comparing the forms of the function f on the overlaps, we see that $f(q) = a + qb$ with the same constants a, b throughout U. $\square$

We will now give a definition of 'regular' for a quaternionic function which is satisfied by a large class of functions and which leads to a development similar to the theory of regular functions of a complex variable.

Definition. A function $f : \mathbb{H} \to \mathbb{H}$ is *left-regular* at $q \in \mathbb{H}$ if it is real-differentiable at q and there exists a quaternion $f_l'(q)$ such that

$$d(dq \wedge dq\, f) = Dq\, f_l'(q). \tag{3.7}$$

It is *right-regular* if there exists a quaternion $f_r'(q)$ such that

$$d(f\, dq \wedge dq) = f_r'(q)\, Dq.$$

Clearly, the theory of left-regular functions will be entirely equivalent to the theory of right-regular functions. For definiteness, we will only consider left-regular functions, which we will call simply *regular*. We write $f_l'(q) = f'(q)$ and call it the *derivative* of f at q.

An application of Stokes's theorem gives the following characterization of the derivative of a regular function as the limit of a difference quotient:

PROPOSITION 2. *Suppose that f is regular at q_0 and continuously differentiable in a neighbourhood of q_0. Then given $\epsilon > 0$, there exists $\delta > 0$ such that if C is a non-degenerate oriented 3-parallelepiped with $q_0 \in C(I^3)$ and $q \in C(I^3) \Rightarrow |q - q_0| < \delta$, then*

$$\left| \left(\int_C Dq \right)^{-1} \left(\int_{\partial C} dq \wedge dq\, f \right) - f'(q_0) \right| < \epsilon.$$

The corresponding characterization of the derivative in terms of the values of the function at a finite number of points is

$$\begin{aligned} f'(q_0) = \lim_{h_1,h_2,h_3 \to 0} [Dq(h_1,h_2,h_3)^{-1}\{(h_1h_2 - h_2h_1)(f(q_0+h_3) - f(q_0)) \\ +(h_2h_3 - h_3h_2)(f(q_0+h_1) - f(q_0)) \\ +(h_3h_1 - h_1h_3)(f(q_0+h_2) - f(q_0))\}]. \end{aligned} \tag{3.8}$$

This is valid if it is understood that h_1, h_2, h_3 are multiples of three fixed linearly independent quaternions, $h_i = t_i H_i$, and the limit is taken as $t_1, t_2, t_3 \to 0$.

By writing (3.7) as

$$dq \wedge dq \wedge df = Dq\, f'(q)$$

and evaluating these trilinear functions with arguments (i, j, k) and $(1, i, j)$, we obtain two equations which give an expression for the derivative as

$$f' = -2\partial_l f = -\frac{\partial f}{\partial t} + i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z} \tag{3.9}$$

and also

PROPOSITION 3. (*the Cauchy-Riemann-Fueter equations*). *A real-differentiable function f is regular at q if and only if $\bar{\partial}_l f = 0$, i.e.*

$$\frac{\partial f}{\partial t} + i\,\frac{\partial f}{\partial x} + j\,\frac{\partial f}{\partial y} + k\,\frac{\partial f}{\partial z} = 0. \tag{3.10}$$

If we write $q = v + jw$, $f(q) = g(v, w) + jh(v, w)$ as in Theorem 1, (3.10) becomes the pair of complex equations

$$\frac{\partial g}{\partial \bar{v}} = \frac{\partial h}{\partial \bar{w}}, \qquad \frac{\partial g}{\partial w} = -\frac{\partial h}{\partial v}, \tag{3.11}$$

which can be seen as a complexification of the Cauchy-Riemann equations for a function of a complex variable.

From Proposition 3 and (2.35) it follows that if f is regular and twice differentiable, then $\Delta f = 0$, i.e. f is harmonic. We will see in the next section that a regular function is necessarily infinitely differentiable, so all regular functions are harmonic.

4. *Cauchy's theorem and the integral formula*. The integral theorems for regular quaternionic functions have as wide a range of validity as those for regular complex functions, which is considerably wider than that of the integral theorems for harmonic functions. Cauchy's theorem holds for any rectifiable contour of integration; the integral formula, which is similar to Poisson's formula in that it gives the values of a function in the interior of a region in terms of its values on the boundary, holds for a general rectifiable boundary, and thus constitutes an explicit solution to the general Dirichlet problem.

The algebraic basis of these theorems is the equation

$$\begin{aligned} d(g\,Dq\,f) &= dg \wedge Dq\,f - g\,Dq \wedge df \\ &= \{(\bar{\partial}_r g)f + g(\bar{\partial}_l f)\}v, \end{aligned} \tag{4.1}$$

which holds for any differentiable functions f and g. Taking $g = 1$ and using Proposition 3, we have:

PROPOSITION 4. *A differentiable function f is regular at q if and only if*

$$Dq \wedge df_q = 0.$$

From this, together with Stokes's theorem, it follows that if f is regular and continuously differentiable in a domain D with differentiable boundary, then

$$\int_{\partial D} Dq\,f = 0.$$

As in complex analysis, however, the conditions on f can be weakened by using Goursat's dissection argument. Applying this to a parallelepiped, we obtain

LEMMA 1. *If f is regular at every point of the 4-parallelepiped C,*

$$\int_{\partial C} Dq\,f = 0. \tag{4.2}$$

The dissection argument can also be used to prove the Cauchy-Fueter integral formula for a parallelepiped:

LEMMA 2. *If f is regular at every point of the positively oriented 4-parallelepiped C, and q_0 is a point in the interior of C,*

$$f(q_0) = \frac{1}{2\pi^2}\int_{\partial C} \frac{(q-q_0)^{-1}}{|q-q_0|^2}\,Dq\,f(q). \tag{4.3}$$

Proof. In (4.1) take $g(q) = \frac{(q-q_0)^{-1}}{|q-q_0|^2} = -\partial_r\left(\frac{1}{|q-q_0|^2}\right)$.

Then g is differentiable except at q_0, and $\bar{\partial}_r g = 0$; hence if f is regular $d(g\,Dq\,f) = 0$ except at q_0. A dissection argument now shows that in the above integral C can be replaced by any smaller 4-parallelepiped C' with $q_0 \in \operatorname{int} C' \subset C$, and since f is continuous at q_0 we can choose C' so small that $f(q)$ can be replaced by $f(q_0)$. Since the 3-form $g\,Dq$ is closed and continuously differentiable in $\mathbb{H} - \{q_0\}$, we can replace $\int_{\partial C'} g\,Dq$ by the integral over a 3-sphere S with centre at q_0, on which

$$Dq = \frac{(q-q_0)}{|q-q_0|}\,dS,$$

where dS is the usual Euclidean volume element on the 3-sphere. Hence

$$\int_{\partial C} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dq\, f(q) = \int_S \frac{dS}{|q-q_0|^3} f(q_0) = 2\pi^2 f(q_0). \qquad \square$$

We will use the following special notation for the function occurring in the integral formula:

$$G(q) = \frac{q^{-1}}{|q|^2}.$$

This function is real-analytic except at the origin; hence in (4.3) the integrand is a continuous function of (q, q_0) in $\partial C \times \operatorname{int} C$ and, for each fixed $q \in \partial C$, a real-analytic function of q_0 in $\operatorname{int} C$. It follows ((**12**), p. 7) that the integral is a real-analytic function of q_0 in $\operatorname{int} C$. Thus we have

THEOREM 1. *A function which is regular in an open set U is real-analytic in U.*

This makes it valid to apply Stokes's theorem and so obtain Cauchy's theorem for the boundary of any differentiable 4-chain. It can be further extended to rectifiable contours, defined as follows:

Definition. Let $C : I^3 \to \mathbb{H}$ be a continuous map of the unit 3-cube into $\mathbb{H}$, and let $P : 0 = s_0 < s_1 < ... < s_p = 1$, $Q : 0 = t_0 < t_1 < ... < t_q = 1$ and

$$R : 0 = u_0 < u_1 < ... < u_r = 1$$

be three partitions of the unit interval I. Define

$$\begin{aligned}\sigma(C; P, Q, R) = \sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\sum_{n=0}^{r-1} Dq(&C(s_{l+1}, t_m, u_n) - C(s_l, t_m, u_n),\\ &C(s_l, t_{m+1}, u_n) - C(s_l, t_m, u_n),\\ &C(s_l, t_m, u_{n+1}) - C(s_l, t_m, u_n)).\end{aligned}$$

C is a *rectifiable 3-cell* if there is a real number M such that $\sigma(C; P, Q, R) < M$ for all partitions P, Q, R. If this is the case the least upper bound of the numbers $(C; P, Q, R)$ is called the *content* of C and denoted by $\sigma(C)$.

Let f and g be quaternion-valued functions defined on $C(I^3)$. We say that $f\,Dq\,g$ is *integrable* over C if the sum

$$\begin{aligned}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\sum_{n=0}^{r-1} f(C(\bar{s}_l, \bar{t}_m, \bar{u}_n))Dq(&C(s_{l+1}, t_m, u_n) - C(s_l, t_m, u_n),\\ &C(s_l, t_{m+1}, u_n) - C(s_l, t_m, u_n),\\ &C(s_l, t_m, u_{n+1}) - C(s_l, t_m, u_n))g(C(\bar{s}, \bar{t}_m, \bar{u}_n)),\end{aligned}$$

where $s_l \le \bar{s}_l \le s_{l+1}$, $t_m \le \bar{t}_m \le t_{m+1}$ and $u_n \le \bar{u}_n \le u_{n+1}$, has a limit in the sense of Riemann-Stieltjes integration as $|P|$, $|Q|$, $|R| \to 0$, where

$$|P| = \max_{0 \le l \le p-1} |s_{l+1} - s_l|$$

measures the coarseness of the partition P. If this limit exists, we denote it by $\int_C f\, Dq\, g$.

We extend these definitions to define rectifiable 3-chains and integrals over rectifiable 3-chains in the usual way.

Just as for rectifiable curves, we can show that $f\, Dq\, g$ is integrable over the 3-chain C if f and g are continuous and C is rectifiable, and

$$\left| \int_C f\, Dq\, g \right| \leq (\max_C |f|)(\max_C |g|)\sigma(C).$$

Furthermore, we have the following weak form of Stokes's theorem:

STOKES'S THEOREM FOR A RECTIFIABLE CONTOUR. *Let C be a rectifiable 3-chain in $\mathbb{H}$ with $\partial C = 0$, and suppose f and g are continuous functions defined in a neighbourhood U of the image of C, and that $f\, Dq\, g = d\omega$ where ω is a 2-form on U. Then*

$$\int_C f\, Dq\, g = 0.$$

The proof proceeds by approximating C by a chain of 3-parallelepipeds with vertices at the points $C_a(s_l, t_m, u_n)$ where C_a is a 3-cell in C and s_l, t_m, u_n are partition points in I. Stokes's theorem holds for this chain of 3-parallelepipeds and we can use the same argument as for rectifiable curves (see, for example, (**13**), p. 103).

We can now give the most general forms of Cauchy's theorem and the integral formula.

THEOREM 2. (*Cauchy's theorem for a rectifiable contour*).

Suppose f is regular in an open set U, and let C be a rectifiable 3-chain which is homologous to 0 in the singular homology of U. Then

$$\int_C Dq\, f = 0.$$

Proof. First we prove the theorem in the case when U is contractible. In this case, since $d(Dq\, f) = 0$ and f is continuously differentiable (by Theorem 1), Poincaré's lemma applies and we have $Dq\, f = d\omega$ for some 2-form ω on D. But $\partial C = 0$, so by Stokes's theorem $\int_C Dq\, f = 0$.

In the general case, suppose $C = \partial C^*$ where C^* is a 4-chain in U. We can dissect C^* as

$$C^* = \sum_n C_n^*,$$

where each C_n^* is a 4-cell lying inside an open ball contained in U and C_n^* is rectifiable. Hence by the first part of the theorem $\int_{\partial C_n^*} Dqf = 0$, and therefore

$$\int_C Dq\, f = \sum_n \int_{\partial C_n^*} Dqf = 0. \qquad \square$$

For the general form of the integral formula, we need an analogue of the notion of the winding number of a curve round a point in the plane. Let q be any quaternion, and let C be a 3-cycle in $\mathbb{H} - \{q\}$. Then C is homologous to $n\, \partial C_0$, where C_0 is a positively oriented 4-parallelepiped in $\mathbb{H} - \{q\}$, and n is an integer (independent of the choice of C_0), which we will call the *wrapping number* of C about q.

THEOREM 3 (*the integral formula for a rectifiable contour*).

Suppose f is regular in an open set U. Let q_0 be a point in U, and let C be a rectifiable 3-chain which is homologous, in the singular homology of $U - \{q_0\}$, to a differentiable 3-chain whose image is ∂B for some ball $B \subset U$. Then

$$\frac{1}{2\pi^2} \int_C \frac{(q - q_0)^{-1}}{|q - q_0|^2} Dq\, f(q) = n f(q_0)$$

where n is the wrapping number of C about q_0.

Many of the standard theorems of complex analysis depend only on Cauchy's integral formula, and so they also hold for quaternionic regular functions. Obvious examples are the maximum-modulus theorem (see, for example, (**14**), p. 165 (first proof)) and Liouville's theorem ((**14**), p. 85 (second proof)). Morera's theorem also holds for quaternionic functions, but in this case the usual proof cannot easily be adapted. It can be proved (**8**) by using a dissection argument to show that if f is continuous in an open set U and satisfies

$$\int_{\partial C} Dq\, f = 0$$

for every 4-parallelepiped C contained in U, then f satisfies the integral formula; and then arguing as for the analyticity of a regular function.

5. *Construction of regular functions.* Regular functions can be constructed from harmonic functions in two ways. First, if f is harmonic then (2.35) shows that $\partial_l f$ is regular. Second, any real-valued harmonic function is, at least locally, the real part of a regular function:

THEOREM 4. *Let u be a real-valued function defined on a star-shaped open set $U \subseteq \mathbb{H}$. If u is harmonic and has continuous second derivatives, there is a regular function f defined on U such that $\operatorname{Re} f = u$.*

Proof. Without loss of generality we may assume that U contains the origin and is star-shaped with respect to it. In this case we will show that the function

$$f(q) = u(q) + 2 \operatorname{Pu} \int_0^1 s^2 \partial_l u(sq) q\, ds \tag{5.1}$$

is regular in U.

Since

$$\begin{aligned} \operatorname{Re} \int_0^1 s^2 \partial_l u(sq) q\, ds &= \frac{1}{2} \int_0^1 s^2 \left\{ t \frac{\partial u}{\partial t}(sq) + x_i \frac{\partial u}{\partial x_i}(sq) \right\} ds \\ &= \frac{1}{2} \int_0^1 s^2 \frac{d}{ds}[u(sq)]\, ds \\ &= \tfrac{1}{2} u(q) - \int_0^1 s u(sq)\, ds, \end{aligned}$$

we can write

$$f(q) = 2 \int_0^1 s^2 \partial_l u(sq) q\, ds + 2 \int_0^1 s u(sq)\, ds. \tag{5.2}$$

Since u and $\partial_l u$ have continuous partial derivatives in U, we can differentiate under the integral sign to obtain, for $q \in U$,

$$\bar{\partial}_l f(q) = 2\int_0^1 s^2 \bar{\partial}_l[\partial_l u(sq)]q\,ds + \int_0^1 s^2\{\partial_l u(sq) + e_i \partial_l u(sq) e_i\}\,ds + 2\int_0^1 s^2 \bar{\partial}_l u(sq)\,ds.$$

But $\bar{\partial}_l[\partial_l u(sq)] = \frac{1}{4}s\Delta u(sq) = 0$ since u is harmonic in U, and

$$\begin{aligned} \partial_l u(sq) + e_i \partial_l u(sq) e_i &= -2\overline{\partial_l u(sq)} \\ &= -2\bar{\partial}_l u(sq) \qquad \text{since } u \text{ is real.} \end{aligned}$$

Hence $\bar{\partial}_l f = 0$ in U and so f is regular. $\square$

If the region U is star-shaped with respect not to the origin but to some other point a, formulae (5.1) and (5.2) must be adapted by changing origin, thus:

$$f(q) = u(q) + 2\,\mathrm{Pu}\int_0^1 s^2 \partial_l u((1-s)a + sq)(q-a)\,ds \tag{5.3}$$

$$= 2\int_0^1 s^2 \partial_l u((1-s)a + sq)(q-a)\,ds + 2\int_0^1 s u((1-s)a + sq)\,ds. \tag{5.4}$$

An example which can be expected to be important is the case of the function

$$u(q) = |q|^{-2}.$$

This is the elementary potential function in four dimensions, as $\log|z|$ is in the complex plane, and so the regular function whose real part is $|q|^{-2}$ is an analogue of the logarithm of a complex variable.

We take U to be the whole of $\mathbb{H}$ except for the origin and the negative real axis. Then U is star-shaped with respect to 1, and $|q|^{-1}$ is harmonic in U. Put

$$u(q) = \frac{1}{|q|^2}, \qquad \partial_l u(q) = -\frac{q^{-1}}{|q|^2}, \qquad a = 1;$$

then (5.3) gives

$$\left.\begin{aligned} f(q) &= -(q\,\mathrm{Pu}\,q)^{-1} - \frac{1}{|\,\mathrm{Pu}\,q|^2}\arg q \quad \text{if } \mathrm{Pu}\,q \neq 0 \\ &= \frac{1}{|q|^2} \quad \text{if } q \text{ is real and positive,} \end{aligned}\right\} \tag{5.5}$$

where

$$\arg q = \log(\mathrm{Un}\,q) = \frac{\mathrm{Pu}\,q}{|\,\mathrm{Pu}\,q|}\tan^{-1}\left(\frac{|\,\mathrm{Pu}\,q|}{\mathrm{Re}\,q}\right), \tag{5.6}$$

which is i times the usual argument in the complex plane generated by q. (In practice the formulae (5.3) and (5.4) are not very convenient to use, and it is easier to obtain (5.5) by solving the equations

$$\nabla.\mathbf{F} = -\frac{2t}{(t^2 + r^2)^2}$$

and

$$\frac{\partial \mathbf{F}}{\partial t} + \nabla \times \mathbf{F} = \frac{2\mathbf{r}}{(t^2 + r^2)^2},$$

where $t = \operatorname{Re} q$, $\mathbf{r} = \operatorname{Pu} q$ and $r = |\mathbf{r}|$—these express the fact that $\mathbf{F} : \mathbb{H} \to P$ is the pure quaternion part of a regular function whose real part is $|q|^{-2}$—and assuming that $\mathbf{F}$ has the form $F(r)\mathbf{r}$.)

We will denote the function (5.5) by $-2L(q)$. The derivative of $L(q)$ can most easily be calculated by writing it in the form

$$L(q) = -\frac{r^2 + te_ix_i}{2r^2(r^2 + t^2)} + \frac{e_ix_i}{2r^3}\tan^{-1}\left(\frac{r}{t}\right); \tag{5.7}$$

the result is

$$\partial_l L(q) = G(q) = \frac{q^{-1}}{|q|^2}. \tag{5.8}$$

Thus $L(q)$ is a primitive for the function occurring in the Cauchy-Fueter integral formula, just as the complex logarithm is a primitive for z^{-1}, the function occurring in Cauchy's integral formula.

Theorem 4 shows that there are as many regular functions of a quaternion variable as there are harmonic functions of four real variables. However, these functions do not include the simple algebraic functions, such as powers of the variable, which occur as analytic functions of a complex variable. Fueter (**4**) also found a method for constructing a regular function of a quaternion variable from an analytic function of a complex variable.

For each $q \in \mathbb{H}$, let $\eta_q : \mathbb{C} \to \mathbb{H}$ be the embedding of the complex numbers in the quaternions such that q is the image of a complex number $\zeta(q)$ lying in the upper half-plane; i.e.

$$\eta_q(x + iy) = x + \frac{\operatorname{Pu} q}{|\operatorname{Pu} q|}y, \tag{5.9}$$

$$\zeta(q) = \operatorname{Re} q + i|\operatorname{Pu} q|. \tag{5.10}$$

Then we have

THEOREM 5. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is analytic in the open set $U \subseteq \mathbb{C}$, and define $\tilde{f} : \mathbb{H} \to \mathbb{H}$ by*

$$\tilde{f}(q) = \eta_q \circ f \circ \zeta(q). \tag{5.11}$$

Then $\Delta\tilde{f}$ is regular in the open set $\zeta^{-1}(U) \subseteq \mathbb{H}$, and its derivative is

$$\partial_l(\Delta\tilde{f}) = \Delta\tilde{f}', \tag{5.12}$$

where f' is the derivative of the complex function f.

For a proof, see (**7**). Note that if we write $f(x + iy) = u(x, y) + iv(x, y)$, $t = \operatorname{Re} q$ and $r = \operatorname{Pu} q$, then

$$\tilde{f}(q) = u(t, r) + \frac{\operatorname{Pu} q}{r}v(t, r), \tag{5.13}$$

$$\Delta\tilde{f}(q) = \frac{2u_2(t, r)}{r} + \frac{2\operatorname{Pu} q}{r}\left\{\frac{v_2(t, r)}{r} - \frac{v(t, r)}{r^2}\right\}, \tag{5.14}$$

where the suffix 2 denotes differentiation with respect to the second argument.

Functions of the form $\tilde{f}$ have been taken as the basis of an alternative theory of functions of a quaternion variable by Cullen (**15**). The following examples are interesting: when

$$f(z) = z^{-1}, \qquad \Delta\tilde{f}(q) = -4G(q); \tag{5.15}$$

when

$$f(z) = \log z, \qquad \Delta\tilde{f}(q) = -4L(q) \tag{5.16}$$

Given a regular function f, other regular functions can be constructed from it by composing it with conformal transformations. The special cases of inversion and rotation are particularly useful:

PROPOSITION 5. (i) *Given a function* $f : \mathbb{H} \to \mathbb{H}$, *let* $If : \mathbb{H} - \{0\} \to \mathbb{H}$ *be the function*

$$If(q) = \frac{q^{-1}}{|q|^2} f(q^{-1}). \tag{5.17}$$

If f *is regular at* q^{-1}, If *is regular at* q.

(ii) *Given a function* $f : \mathbb{H} \to \mathbb{H}$ *and constant quaternions* a, b, *let* $M(a,b)f$ *be the function*

$$[M(a,b)f](q) = bf(a^{-1}qb). \tag{5.18}$$

If f *is regular at* $a^{-1}qb$, $M(a,b)f$ *is regular at* q.

Proof. (i) By Proposition 4 it is sufficient to show that

$$Dq \wedge d(If)_q = 0.$$

Now $If = G(f \circ \imath)$, where $G(q) = q^{-1}/|q|^2$ and $\imath : \mathbb{H} - \{0\} \to \mathbb{H}$ is the inversion $q \mapsto q^{-1}$. Hence

$$\begin{aligned} Dq \wedge d(If)_q &= Dq \wedge dG_q f(q^{-1}) + Dq \wedge G(q) d(f \circ \imath)_q \\ &= Dq\, G(q) \wedge \imath_q^* df_{q^{-1}} \end{aligned}$$

since G is regular at $q \neq 0$. But

$$\begin{aligned} \imath_q^* Dq(h_1, h_2, h_3) &= Dq(-q^{-1}h_1q^{-1}, -q^{-1}h_2q^{-1}, -q^{-1}h_3q^{-1}) \\ &= -\frac{q^{-1}}{|q|^4} Dq(h_1, h_2, h_3) q^{-1} \end{aligned}$$

by (2.31). Thus

$$Dq\, G(q) = -|q|^2 q \imath_q^* Dq$$

and so

$$\begin{aligned} Dq \wedge d(If)_q &= -|q|^2 q \imath_q^* (Dq \wedge df_{q^{-1}}) \\ &= 0 \end{aligned}$$

if f is regular at q^{-1}.

(ii) Let $\mu : \mathbb{H} \to \mathbb{H}$ be the map $q \mapsto aqb$. Then by (2.31)

$$\mu^* Dq = |a|^2 |b|^2 a\, Dq\, b$$

and so

$$\begin{aligned} Dq \wedge d[M(a,b)f]_q &= Dq \wedge b\mu_q^* df_{\mu(q)} \\ &= |a|^{-2}|b|^{-2}a^{-1}(\mu_q^* Dq)b^{-1} \wedge b\mu_q^* df_{\mu(q)} \\ &= |a|^{-2}|b|^{-2}a^{-1}\mu_q^*(Dq \wedge df_{\mu(q)}) \\ &= 0 \end{aligned}$$

if f is regular at $\mu(q)$. It follows from Proposition 4 that $M(a,b)f$ is regular at q. $\square$

The general conformal transformation of the one-point compactification of $\mathbb{H}$ is of the form

$$\nu(q) = (aq+b)(cq+d)^{-1} \tag{5.19}$$

with $a^{-1}b \neq c^{-1}d$. Such a transformation ((**16**), p. 312) is the product of a sequence of transformations of the types considered in Proposition 5, together with translations $q \mapsto q + a$ (which clearly preserve regularity). The corresponding transformation of regular functions is as follows:

THEOREM 6. *Given a function $f : \mathbb{H} \to \mathbb{H}$ and a conformal transformation ν as in (5.19), let $M(\nu)f$ be the function*

$$[M(\nu)f](q) = \frac{1}{|b - ac^{-1}d|^2}\frac{(cq+d)^{-1}}{|cq+d|^2} f(\nu(q)).$$

If f is regular at $\nu(q)$, $M(\nu)f$ is regular at q.

6. *Homogeneous regular functions*. In this section we will study the relations between regular polynomials, harmonic polynomials and harmonic analysis on the group S of unit quaternions, which is to quaternionic analysis what Fourier analysis is to complex analysis.

The basic Fourier functions $e^{in\theta}$ and $e^{-in\theta}$, regarded as functions on the unit circle in the complex plane, each have two extensions to harmonic functions on $\mathbb{C} - \{0\}$; thus we have the four functions z^n, $\bar{z}^n$, z^{-n} and $\bar{z}^{-n}$. The requirement of analyticity picks out half of these, namely z^n and z^{-n}. In the same way the basic harmonic functions on S, namely the matrix elements of unitary irreducible representations of S, each have two extensions to harmonic functions on $\mathbb{H} - \{0\}$, one with a negative degree of homogeneity and one with a positive degree. We will see that the space of functions belonging to a particular unitary representation, corresponding to the space of combinations of $e^{in\theta}$ and $e^{-in\theta}$ for a particular value of n, can be decomposed into two complementary subspaces; one (like $e^{in\theta}$) gives a regular function on $\mathbb{H} - \{0\}$ when multiplied by a positive power of $|q|$, the other (like $e^{-in\theta}$) has to be multiplied by a negative power of $|q|$.

Let U_n be the set of functions $f : \mathbb{H} - \{0\} \to \mathbb{H}$ which are regular and homogeneous of degree n over $\mathbb{R}$, i.e.

$$f(\alpha q) = \alpha^n f(q) \qquad \text{for } \alpha \in R.$$

Removing the origin from the domain of f makes it possible to consider both positive and negative n (the alternative procedure of adding a point at infinity to $\mathbb{H}$ has disadvantages, since regular polynomials do not necessarily admit a continuous extension to

$\mathbb{H} \cup \{\infty\} \cong S^4$). Let W_n be the set of functions $f : \mathbb{H} - \{0\} \to \mathbb{H}$ which are harmonic and homogeneous of degree n over R. Then U_n and W_n are right vector spaces over $\mathbb{H}$ (with pointwise addition and scalar multiplication) and since every regular function is harmonic, we have $U_n \subseteq W_n$.

Functions in U_n and W_n can be studied by means of their restriction to the unit sphere $S = \{q : |q| = 1\}$. Let

$$\tilde{U}_n = \{f|S : f \in U_n\}, \qquad \tilde{W}_n = \{f|S : f \in W_n\};$$

then U_n and $\tilde{U}_n$ are isomorphic (as quaternionic vector spaces) by virtue of the correspondence

$$f \in U_n \Leftrightarrow \tilde{f} \in \tilde{U}_n, \qquad \text{where} \qquad f(q) = r^n \tilde{f}(u), \tag{6.1}$$

using the notation $r = |q| \in \mathbb{R}$, $u = q/|q| \in S$.

Similarly W_n and $\tilde{W}_n$ are isomorphic.

In order to express the Cauchy-Riemann-Fueter equations in a form adapted to the polar decomposition $q = ru$, we introduce the following vector fields $X_0, ..., X_3$ on $\mathbb{H} - \{0\}$:

$$X_0 f = \frac{d}{d\theta}[f(qe^\theta)]_{\theta=0}, \tag{6.2}$$

$$X_i f = \frac{d}{d\theta} f[q \exp(e_i \theta)]_{\theta=0} = \frac{d}{d\theta} f[q(\cos\theta + e_i \sin\theta)]_{\theta=0} \quad (i = 1, 2, 3). \tag{6.3}$$

These fields form a basis for the real vector space of left-invariant vector fields on the multiplicative group of $\mathbb{H}$, and they are related to the Cartesian vector fields $\partial/\partial t$, $\partial/\partial x_i$ by

$$X_0 = t\,\frac{\partial}{\partial t} + x_i\,\frac{\partial}{\partial x_i}, \tag{6.4}$$

$$X_i = -x_i\,\frac{\partial}{\partial t} + t\,\frac{\partial}{\partial x_i} - \epsilon_{ijk}\,x_j\,\frac{\partial}{\partial x_k}, \tag{6.5}$$

$$\frac{\partial}{\partial t} = \frac{1}{r^2}(tX_0 - x_i X_i), \tag{6.6}$$

$$\frac{\partial}{\partial x_i} = \frac{1}{r^2}(\epsilon_{ijk} x_j X_k + tX_i + x_i X_0). \tag{6.7}$$

Their Lie brackets are

$$[X_0, X_i] = 0, \tag{6.8}$$

$$[X_i, X_j] = 2\epsilon_{ijk} X_k. \tag{6.9}$$

Using (6.6) and (6.7) the differential operators $\bar{\partial}_l$ and Δ can be calculated in terms of X_0 and X_i. The result is

$$\bar{\partial}_l = \tfrac{1}{2}\bar{q}^{-1}(X_0 + e_i X_i), \tag{6.10}$$

$$\Delta = \frac{1}{r^2}\left\{X_i X_i + X_0(X_0 + 2)\right\}. \tag{6.11}$$

The following facts about the space of harmonic functions W_n are well known (and follow from (6.11); see, for example, (**17**), p. 71):

PROPOSITION 6. (i) $\tilde{W}_n \cong \tilde{W}_{-n-2}$. (ii) $\dim W_n = (n+1)^2$. (iii) *The elements of W_n are polynomials in q.*

We can now obtain the basic facts about the spaces U_n of regular functions:

THEOREM 7. (i) $\tilde{W}_n = \tilde{U}_n \oplus \tilde{U}_{-n-2}$. (ii) $U_n \cong U_{-n-3}$. (iii) $\dim U_n = \frac{1}{2}(n+1)(n+2)$.

Proof. (i) Equation (6.10) shows that the elements of U_n, which satisfy $X_0 f = nf$ and $\bar{\partial}_l f = 0$, are eigenfunctions of $\Omega = e_i X_i$ with eigenvalue $-n$. Since the vector fields X_i are tangential to the sphere S, Ω can be considered as an operator on $\tilde{W}_n$, and $\tilde{U}_n$ consists of the eigenfunctions of Ω with eigenvalue $-n$. Using (6.9), it can be shown that

$$\Omega^2 - 2\Omega + X_i X_i = 0.$$

Hence

$$\begin{aligned} \tilde{f} \in \tilde{W}_n &\Rightarrow \Delta(r^n f) = 0 \\ &\Rightarrow X_i X_i f = -n(n+2)f \\ &\Rightarrow (\Omega - n - 2)(\Omega + n)f = 0. \end{aligned}$$

It follows that $\tilde{W}_n$ is the direct sum of the eigenspaces of $\tilde{\Omega}$ with eigenvalues $-n$ and $n+2$ (these are vector subspaces of $\tilde{W}_n$ since the eigenvalues are real), i.e.

$$\tilde{W}_n = \tilde{U}_n \oplus \tilde{U}_{-n-2}.$$

(ii) It follows from Proposition 5(i) that the mapping I is an isomorphism between U_n and U_{-n-3}.

(iii) Let $d_n = \dim U_n$. By (i) and Proposition 6(ii),

$$d_n + d_{-n-2} = (n+1)^2$$

and by (ii),

$$d_{-n-2} = d_{n-1}.$$

Thus

$$d_n + d_{n-1} = (n+1)^2.$$

The solution of this recurrence relation, with $d_0 = 1$, is

$$d_n = \tfrac{1}{2}(n+1)(n+2). \qquad \square$$

There is a relation between Proposition 5 (ii) and the fact that homogeneous regular functions are eigenfunctions of Ω. Proposition 5 (ii) refers to a representation M of the group $\mathbb{H}^\times \times \mathbb{H}^\times$ defined on the space of real-differentiable functions $f : \mathbb{H} - \{0\} \to \mathbb{H}$ by

$$[M(a,b)f](q) = b\, f(a^{-1}qb).$$

Restricting to the subgroup $\{(a,b) : |a| = |b| = 1\}$, which is isomorphic to

$$SU(2) \times SU(2),$$

we obtain a representation of $SU(2) \times SU(2)$. Since the map $q \mapsto aqb$ is a rotation when $|a| = |b| = 1$, the set W of harmonic functions is an invariant subspace under this representation. Now $W = \mathbb{H} \otimes_{\mathbb{C}} W^c$, where W^c is the set of complex-valued harmonic functions, and the representation of $SU(2) \times SU(2)$ can be written as

$$M(a,b)(q \otimes f) = (bq) \otimes R(a,b)f$$

where R denotes the quasi-regular representation corresponding to the action $q \mapsto aqb^{-1}$ of $SU(2) \times SU(2)$ on $\mathbb{H} - \{0\}$:

$$[R(a,b)f]q = f(a^{-1}qb).$$

Thus $M|W$ is the tensor product of the representations $D^0 \times D^1$ and $R|W^c$ of

$$SU(2) \times SU(2),$$

where D^n denotes the $(n+1)$-dimensional complex representation of $SU(2)$. The isotypic components of $R|W^c$ are the homogeneous subspaces W_n^c, on which R acts irreducibly as $D^n \times D^n$; thus W_n is an invariant subspace under the representation M, and $M|W_n$ is the tensor product $(D^0 \times D^1) \otimes (D^n \times D^n)$. W_n therefore has two invariant subspaces, on which M acts as the irreducible representations $D^n \times D^{n+1}$ and

$$D^n \times D^{n-1}.$$

These subspaces are the eigenspaces of Ω. To see this, restrict attention to the second factor in $SU(2) \times SU(2)$; we have the representation

$$M'(b)(q \otimes f) = M(1,b)(q \otimes f) = [D^1(b)q] \otimes [R(1,b)f]$$

where $D^1(b)q = bq$. The infinitesimal generators of the representation $R(1,b)$ are the differential operators X_i; the infinitesimal generators of $D^1(b)$ are e_i (by which we mean left multiplication by e_i). Hence the infinitesimal operators of the tensor product M' are $e_i + X_i$. The isotypic components of W are the eigenspaces of the Casimir operator

$$(e_i + X_i)(e_i + X_i) = e_ie_i + X_iX_i + 2\Omega.$$

But $e_ie_i = -3$, and $X_iX_i = -n(n+2)$ on W_n; hence

$$(e_i + X_i)(e_i + X_i) = 2\Omega - n^2 - 2n - 3.$$

and so the isotypic components of W_n for the representation M' are the eigenspaces of Ω. U_n, the space of homogeneous regular functions of degree n, has eigenvalue $-n$ for Ω, and so $M'|U_n$ is the representation D^{n+1} of $SU(2)$.

Similar considerations lead to the following fact:

PROPOSITION 7. *If f is regular, qf is harmonic.*

The representation M of $SU(2) \times SU(2)$ can also be used to find a basis of regular polynomials. It belongs to a class of induced representations which is studied in (**18**), where a procedure is given for splitting the representation into irreducible components and finding a basis for each component. Rather than give a rigorous heuristic derivation by following this procedure, which is not very enlightening in this case, we will state the result and then verify that it is a basis.

Since the functions to be considered involve a number of factorials, we introduce the notation

$$\begin{aligned} z^{[n]} &= \frac{z^n}{n!} \quad \text{if } n \leq 0 \\ &= 0 \quad \text{if } n < 0 \end{aligned}$$

for a complex variable z. This notation allows the convenient formulae

$$\frac{d}{dz} z^{[n]} = z^{[n-1]}, \tag{6.12}$$

$$(z_1 + z_2)^{[n]} = \sum_r z_1^{[r]} z_2^{[n-r]} \tag{6.13}$$

where the sum is over all integers r.

The representation D^n of $S \cong SU(2)$ acts on the space of homogeneous polynomials of degree n in two complex variables by

$$[D^n(u)f](z_1, z_2) = f(z_1', z_2'),$$

where

$$z_1' + jz_2' = u^{-1}(z_1 + jz_2).$$

Writing $u = v + jw$ where $v, w \in \mathbb{C}$ and $|v|^2 + |w|^2 = 1$, we have

$$z_1' = \bar{v}z_1 + \bar{w}z_2, \qquad z_2' = -wz_1 + vz_2.$$

Hence the matrix elements of $D^n(u)$ relative to the basis $f_k(z_1, z_2) = z_1^{[k]} z_2^{[n-k]}$ are

$$D^n_{kl}(u) = (-)^n k!(n-k)! P^n_{kl}(u)$$

where

$$P^n_{kl}(v + jw) = \sum_r (-)^r v^{[n-k-l+r]} \bar{v}^{[r]} w^{[k-r]} \bar{w}^{[l-r]} \tag{6.14}$$

The functions $P^n_{kl}(q)$ are defined for all quaternions $q = v + jw$ and for all integers k, l, n, but they are identically zero unless $0 \leq k$, $l \leq n$.

PROPOSITION 8. *As a right vector space over* $\mathbb{H}$, U_n *has the basis*

$$Q^n_{kl}(q) = P^n_{kl}(q) - jP^n_{k-1,l}(q) \qquad (0 \leq k \leq l \leq n).$$

Proof. Using (6.14), it is easy to verify that Q^n_{kl} satisfies the Cauchy-Riemann-Fueter equations in the form (cf. 3.11)

$$\frac{\partial P^n_{kl}}{\partial \bar{v}} = -\frac{\partial P^n_{k-1,l}}{\partial \bar{w}}, \qquad \frac{\partial P^n_{kl}}{\partial w} = \frac{\partial P^n_{k-1,l}}{\partial v}.$$

Since the functions D^n_{kl} are independent over $\mathbb{C}$ as functions on S for $0 \leq k$, $l \leq n$, the functions P^n_{kl} are independent over $\mathbb{C}$ as functions on $\mathbb{H}$ for $0 \leq k$, $l \leq n$. It follows that the functions Q^n_{kl} $(0 \leq k \leq n+1,\ 0 \leq l \leq n)$ are independent over $\mathbb{C}$ and therefore span a right vector space over $\mathbb{H}$ of dimension at least $\frac{1}{2}(n+1)(n+2)$. Since this space is a subspace of U_n which has dimension $\frac{1}{2}(n+1)(n+2)$, the Q^n_{kl} span U_n.

Since $zj = j\bar{z}$ for any $z \in \mathbb{C}$, it can be seen from the definition (6.14) that

$$P^n_{kl}\, j = j\, P^n_{n-k,n-l}$$

and therefore

$$Q^n_{kl}\, j = Q^n_{n-k+1,n-l}.$$

Thus U_n is spanned by the Q^n_{kl} ($0 \le k \le l \le n$), which therefore form a basis for U_n. $\square$

Another basis for U_n will be given in the next section.

We conclude this section by studying the quaternionic derivative ∂_l. Since ∂_l is a linear map from U_n into U_{n-1} and $\dim U_n > \dim U_{n-1}$, ∂_l must have a large kernel and so we cannot conclude from $\partial_l f = 0$ that f is constant. However, although the result is far from unique, it is possible to integrate regular polynomials:

THEOREM 8. *Every regular polynomial has a primitive, i.e.* ∂_l *maps* U_n *onto* U_{n-1} *if* $n > 0$.

Proof. Suppose $f \in U_n$ is such that $\partial_l f = 0$. Then

$$\frac{\partial f}{\partial t} = e_i\,\frac{\partial f}{\partial x_i} = 0.$$

Thus f can be regarded as a function on the space P of pure imaginary quaternions. Using vector notation for elements of P and writing $f = f_0 + \mathbf{f}$ with $f_0 \in \mathbb{R}$, $\mathbf{f} \in P$, the condition $e_i\,\partial f/\partial x_i = 0$ becomes

$$\nabla f_0 + \nabla \times \mathbf{f} = 0, \qquad \nabla.\mathbf{f} = 0.$$

If $n \ge 0$, we can define $f(0)$ so that these hold throughout P, and so there exists a function $\mathbf{F} : P \to P$ such that

$$\mathbf{f} = \nabla \times \mathbf{F}, \qquad f_0 = -\nabla.\mathbf{F},$$

i.e.

$$f = e_i\,\frac{\partial \mathbf{F}}{\partial x_i}.$$

Then $\mathbf{F}$ is harmonic, i.e. $\nabla^2\mathbf{F} = 0$.

Let T_n be the right quaternionic vector space of functions $\mathbf{F} : P \to \mathbb{H}$ which are homogeneous of degree n and satisfy $\nabla^2\mathbf{F} = 0$; then $\dim T_n = 2n+1$. Let K_n be the subspace of T_n consisting of functions satisfying $e_i\,\partial\mathbf{F}/\partial x_i = 0$; then $K_n = \ker \partial_l \subset U_n$. The above shows that $e_i\,\partial/\partial x_i : T_{n+1} \to T_n$ maps T_{n+1} onto K_n; its kernel is K_{n+1} and so

$$\dim K_n + \dim K_{n+1} = \dim T_{n+1} = 2n+3.$$

The solution of this recurrence relation, with $\dim K_0 = 1$, is $\dim K_n = n+1$. But

$$\dim U_n - \dim U_{n-1} = \tfrac{1}{2}(n+1)(n+2) - \tfrac{1}{2}n(n+1) = n+1.$$

It follows that ∂_l maps U_n onto U_{n-1}. $\square$

THEOREM 9. *If* $n < 0$*, the map* $\partial_l : U_n \to U_{n-1}$ *is one-to-one.*

Proof. We introduce the following inner product between functions defined on the unit sphere S:

$$\langle f, g \rangle = \int_S \overline{f(u)} g(u) du$$

where du denotes Haar measure on the group S, normalized so that

$$\int_S du = \tfrac{1}{2}\pi^2.$$

For functions defined on $\mathbb{H}$, we can write this as

$$\langle f, g \rangle = \int_S \overline{f(q)} q^{-1} Dq \, g(q)$$

As a map: $U_n \times U_n \to H$, this is antilinear in the first variable and linear in the second, i.e.

$$\langle fa, gb \rangle = \bar{a} \langle f, g \rangle b \quad \text{for all } a, b \in \mathbb{H}$$

and is non-degenerate since $\langle f, f \rangle = 0 \Leftrightarrow f = 0$.

Now let $f \in U_n$, $g \in U_{-n-2}$ and let I denote the map: $U_n \to U_{-n-3}$ defined in Proposition 5(i). Then

$$\begin{aligned}\langle g, I\partial_l f \rangle &= \int_S \overline{g(q)} q^{-1} Dq \, q^{-1} \partial_l f(q^{-1}) \\ &= -\int_S \overline{g(q)} \imath^* (Dq \, \partial_l f),\end{aligned}$$

where $\imath$ denotes the map $q \mapsto q^{-1}$ and we have used the fact that $\imath^* Dq = -q^{-1} Dq \, q^{-1}$ for $q \in S$. Since f is regular, $Dq \, \partial_l f = \frac{1}{2} d(dq \wedge dq \, f)$ and so

$$\begin{aligned}\langle g, I\partial_l f \rangle &= -\frac{1}{2} \int_S \overline{g(q)} d[\imath^*(dq \wedge dq \, f)] \\ &= \frac{1}{2} \int_S \overline{dg} \wedge \imath^*(dq \wedge dq \, f) \quad \text{since } \partial S = 0.\end{aligned}$$

On S, the inversion $\imath$ coincides with quaternion conjugation; hence $\imath^* dq = d\bar{q}$ and therefore

$$\begin{aligned}\langle g, I\partial_l f \rangle &= \frac{1}{2} \int_S d\bar{g} \wedge d\bar{q} \wedge d\bar{q} f(q^{-1}) \\ &= \frac{1}{2} \int_S \overline{dq \wedge dq \wedge dg} f(q^{-1}) \\ &= \int_S \overline{Dq \wedge \partial_l g(q)} f(q^{-1})\end{aligned}$$

since g is regular. Since conjugation is an orthogonal transformation with determinant -1, $Dq(\bar{h}_1, \bar{h}_2, \bar{h}_3) = -\overline{Dq(h_1, h_2, h_3)}$; hence, because conjugation is the same as inversion on S,

$$\overline{Dq} = -\imath^* Dq = q^{-1} Dq \, q^{-1}.$$

Thus

$$\begin{aligned}\langle g, I\partial_l f\rangle &= \int_S \overline{\partial_l g(q)} q^{-1} Dq\, q^{-1} f(q^{-1})\\ &= \langle \partial_l g, If\rangle.\end{aligned}$$

But I is an isomorphism, the inner product is non-degenerate on U_{-n-2}, and ∂_l maps U_{-n-2} onto U_{-n-3} if $n \leq -3$; it follows that $\partial_l : U_n \to U_{n-1}$ is one-to-one. $\square$

In the missing cases $n = -1$ and $n = -2$, Theorems 8 and 9 are both true trivially, since $U_{-1} = U_{-2} = \{0\}$.

7. *Regular power series*. The power series representing a regular function, and the Laurent series representing a function with an isolated singularity, are most naturally expressed in terms of certain special homogeneous functions.

Let ν be an unordered set of n integers $\{i_1, ..., i_n\}$ with $1 \leq i_r \leq 3$; ν can also be specified by three integers n_1, n_2, n_3 with $n_1 + n_2 + n_3 = n$, where n_1 is the number of 1's in ν, n_2 the number of 2's and n_3 the number of 3's, and we will write $\nu = [n_1 n_2 n_3]$. There are $\frac{1}{2}(n+1)(n+2)$ such sets ν; we will denote the set of all of them by σ_n. They are to be used as labels; when $n = 0$, so that $\nu = \emptyset$, we use the suffix 0 instead of $\emptyset$. We write ∂_ν for the nth order differential operator

$$\partial_\nu = \frac{\partial^n}{\partial x_{i_1} ... \partial x_{i_n}} = \frac{\partial^n}{\partial x^{n_1} \partial y^{n_2} \partial z^{n_3}}.$$

The functions in question are

$$G_\nu(q) = \partial_\nu G(q) \tag{7.1}$$

and

$$P_\nu(q) = \frac{1}{n!} \sum (te_{i_1} - x_{i_1}) ... (te_{i_n} - x_{i_n}) \tag{7.2}$$

where the sum is over all $n!/(n_1!n_2!n_3!)$ different orderings of n_1 1's, n_2 2's and n_3 3's. Then P_ν is homogeneous of degree n and G_ν is homogeneous of degree $-n-3$.

As in the previous section, U_n will denote the right quaternionic vector space of homogeneous regular functions of degree n.

PROPOSITION 9. *The polynomials $P_\nu (\nu \in \sigma_n)$ are regular and form a basis for U_n.*

Proof. (**17**) Let f be a regular homogeneous polynomial of degree n. Since f is regular

$$\frac{\partial f}{\partial t} + \sum_i e_i \frac{\partial f}{\partial x_i} = 0$$

and since it is homogeneous,

$$t \frac{\partial f}{\partial t} + \sum_i x_i \frac{\partial f}{\partial x_i} = nf(q).$$

Hence

$$nf(q) = \sum_i (x_i - te_i) \frac{\partial f}{\partial x_i}.$$

But $\partial f/\partial x_i$ is regular and homogeneous of degree $n-1$, so we can repeat the argument; after n steps we obtain

$$\begin{aligned}
f(q) &= \frac{1}{n!}\sum_{i_1\ldots i_n}(x_{i_1}-te_{i_1})\ldots(x_{i_n}-te_{i_n})\frac{\partial^n f}{\partial x_{i_1}\ldots\partial x_{i_n}} \\
&= \sum_{\nu\in\sigma_n}(-1)^n P_\nu(q)\partial_\nu f(q).
\end{aligned}$$

Since f is a polynomial, $\partial_\nu f$ is a constant; thus any regular homogeneous polynomial is a linear combination of the P_ν. Let V_n be the right vector space spanned by the P_ν. By proposition 6 (iii), the elements of U_n are polynomials, so $U_n \subseteq V_n$; but

$$\dim V_n \leq \tfrac{1}{2}(n+1)(n+2) = \dim U_n$$

by Theorem 7(iii). Hence $V_n = U_n$. $\square$

The mirror image of this argument proves that the P_ν are also right-regular.

Just as for a complex variable, we have

$$(1-q)^{-1} = \sum_{n=0}^{\infty} q^n$$

for $|q| < 1$; the series converges absolutely and uniformly in any ball $|q| \leq r$ with $r < 1$. This gives rise to an expansion of $G(p-q)$ in powers of $p^{-1}q$; identifying it with the Taylor series of G about p, we obtain

PROPOSITION 10. *The expansions*

$$\begin{aligned}
G(p-q) &= \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n} P_\nu(q)G_\nu(p) \\
&= \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n} G_\nu(p)P_\nu(q)
\end{aligned}$$

are valid for $|q| < |p|$; the series converge uniformly in any region $\{(p,q) : |q| \leq r|p|\}$ of $\mathbb{H}^2$ with $r < 1$.

Now the same arguments as in complex analysis give:

THEOREM 10. *Suppose f is regular in a neighbourhood of 0. Then there is a ball B with centre 0 in which $f(g)$ is represented by a uniformly convergent series*

$$f(q) = \sum_{n=0}\sum_{\nu\in\sigma_n} P_\nu(q)q_\nu,$$

where the coefficients a_ν are given by

$$\begin{aligned}
a_\nu &= \frac{1}{2\pi^2}\int_{\partial B} G_\nu(q)\, Dq\, f(q) \\
&= (-1)^n\partial_\nu f(0).
\end{aligned}$$

COROLLARY.
$$\frac{1}{2\pi^2}\int_S G_\mu(q)\,Dq\,P_\nu(q) = \delta_{\mu\nu},$$
where S is any sphere containing the origin.

THEOREM 11 (*the Laurent series*). *Suppose f is regular in an open set U except possibly at $q_0 \in U$. Then there is a neighbourhood N of q_0 such that if $q \in N$ and $q \neq q_0$, $f(q)$ can be represented by a series*
$$f(q) = \sum_{n=0}^{\infty}\sum_{\nu\in\sigma_n}\{P_\nu(q-q_0)a_\nu + G_\nu(q-q_0)b_\nu\}$$
which converges uniformly in any hollow ball
$$\{q : r \leq |q-q_0| \leq R\}, \qquad \text{with } r > 0\text{, which lies inside } N.$$
The coefficients a_ν and b_ν are given by
$$a_\nu = \frac{1}{2\pi^2}\int_C G_\nu(q-q_0)\,Dq\,f(q),$$
$$b_\nu = \frac{1}{2\pi^2}\int_C P_\nu(q-q_0)\,Dq\,f(q),$$
where C is any closed 3-chain in $U - \{q_0\}$ which is homologous to ∂B for some ball with $q_0 \in B \subset U$ (so that C has wrapping number 1 about q_0).

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