1 Classical Electrodynamics

Maxwell speculated that someday quaternions would be useful in the analysis of electromagnetism. Hopefully after a 130 year wait, in this notebook we can begin that process. This approach relies on a judicious use of commutators and anticommutators.

The Maxwell Equations

The Maxwell equations are formed from a combinations of commutators and anticommutators of the differential operator and the electric and magnetic fields E and B respectively (for isolated charges in a vacuum.

$$\text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(0, \vec{B}\right)\right) + \text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(0, \vec{E}\right)\right) =$$

$$\left(-\vec{\nabla} \cdot \vec{B},\ \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t}\right) = \left(0, \vec{0}\right)$$

$$\text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(0, \vec{B}\right)\right) - \text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(0, \vec{E}\right)\right) =$$

$$\left(\vec{\nabla} \cdot \vec{E},\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t}\right) = 4\pi\left(\rho, \vec{J}\right)$$

$$\text{where even}(A, B) = \frac{AB + BA}{2},\ \text{odd}(A, B) = \frac{AB - BA}{2}$$

The first quaternion equation embodies the homogeneous Maxwell equations. The scalar term says that there are no magnetic monopoles. The vector term is Faraday's law. The second quaternion equation is the source term. The scalar equation is Gauss' law. The vector term is Ampere's law, with Maxwell's correction.

The 4-Potential A

The electric and magnetic fields are often viewed as arising from the same 4-potential A. These can also be expressed easily using quaternions.

$$e = \text{vector}\left(\text{even}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right)\right) = \left(0, -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right)$$

$$B = \text{odd}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right) = \left(0, \vec{\nabla} \times \vec{A}\right)$$

The electric field E is the vector part of the anticommutator of the conjugates of the differential operator and the 4-potential. The magnetic field B involves the commutator.

These forms can be directly placed into the Maxwell equations.

$$\text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{odd}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right)\right) +$$

$$\text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{vector}\left(\text{even}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right)\right)\right) =$$

$$= \left(-\vec{\nabla} \cdot \vec{\nabla} \times \vec{A},\ \frac{\partial \vec{\nabla} \times \vec{A}}{\partial t} - \vec{\nabla} \times \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \times \vec{\nabla}\phi\right) = \left(-\vec{\nabla} \cdot \vec{B},\ \frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E}\right) = \left(0, \vec{0}\right)$$

$$\text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{odd}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, \vec{A}\right)\right)\right) -$$

$$\text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{vector}\left(\text{even}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right)\right)\right) =$$

$$= \left(- \vec{\nabla} \cdot \vec{\nabla} \phi - \vec{\nabla} \cdot \frac{\partial \vec{A}}{\partial t}, \ \vec{\nabla} \times \vec{\nabla} \times \vec{A} + \frac{\partial^2 \vec{A}}{\partial t^2} + \frac{\partial \vec{\nabla} \phi}{\partial t} \right) =$$

$$\left(\vec{\nabla} \cdot \vec{E} \ , \ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \right) = 4\pi \left(\rho, \ \vec{J} \right)$$

The homogeneous terms are formed from the sum of both orders of the commutator and anticommutator. The source terms arise from the difference of two commutators and two anticommutators.

The Lorentz Force

The Lorentz force is generated similarly to the source term of the Maxwell equations, but there a small game required to get the signs correct for the 4-force.

$$\text{odd}\left(\left(\gamma, \ \gamma \vec{\beta} \right), \ \left(0, \ \vec{B} \right) \right) - \text{even}\left(\left(-\gamma, \ \gamma \vec{\beta} \right), \ \left(0, \ \vec{E} \right) \right) = \left(\gamma \vec{\beta} \cdot \vec{E} \ , \gamma \vec{E} + \gamma \vec{\beta} \times \vec{B} \right)$$

This is the covariant form of the Lorentz force. The additional minus sign required may be a convention handed down through the ages.

Conservation Laws

The continuity equation–conservation of charge–is formed by applying the conjugate of the differential operator to the source terms of the Maxwell equations.

$$\text{scalar}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla} \right) \left(\vec{\nabla} \cdot \vec{E}, \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \right) \right) = \left(\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{E} - \vec{\nabla} \cdot \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \cdot \vec{\nabla} \times \vec{B}, \ 0 \right) =$$

$$= \text{scalar}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla} \right), 4\pi \left(\rho, \ \vec{J} \right) \right) = 4\pi \left(\vec{E} \cdot \vec{J} + \frac{\partial \rho}{\partial t}, \ 0 \right)$$

The upper is zero, so the dot product of the E field and the current density plus the rate of change of the charge density must equal zero. That means that charge is conserved.

Poynting's theorem for energy conservation is formed in a very similar way, except that the conjugate of electric field is used instead of the conjugate of the differential operator.

$$\text{scalar}\left(\left(0, -\vec{E} \right) \left(\vec{\nabla} \cdot \vec{E}, \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} \right) \right) = \left(\vec{E} \cdot \vec{\nabla} \times \vec{B} - \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}, \ 0 \right)$$

$$= \text{scalar}\left(\left(0, -\vec{E} \right), 4\pi \left(\rho, \ \vec{J} \right) \right) = 4\pi \left(\vec{E} \cdot \vec{J}, \ 0 \right)$$

Additional vector identities are required before the final form is reached.

$$\vec{E} \cdot \left(\vec{\nabla} \times \vec{B} \right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{E} \right) + \vec{\nabla} \cdot \left(\vec{B} \times \vec{E} \right)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E} . \frac{\partial \vec{E}}{\partial t} = \frac{1}{2} \left(\frac{\partial \vec{E}}{\partial t} \right)^2$$

$$\vec{B} \cdot \frac{\partial \vec{B}}{\partial t} = \frac{1}{2} \left(\frac{\partial \vec{B}}{\partial t} \right)^2$$

Use these equations to simplify to the following.

$$4\pi \left(\vec{E} \cdot \vec{J}, \ 0 \right) = \left(- \vec{\nabla} \cdot \left(\vec{E} \times \vec{B} \right) - \frac{1}{2} \left(\frac{\partial \vec{E}}{\partial t} \right)^2 - \frac{1}{2} \left(\frac{\partial \vec{B}}{\partial t} \right)^2, \ 0 \right)$$

This is Poynting's equation.

Implications

The foundations of classical electrodynamics are the Maxwell equations, the Lorentz force, and the conservation laws. In this notebook, these basic elements have been written as quaternion equations, exploiting the actions of commutators and anticommutators. There is an interesting link between the E field and a differential operator for generating conservation laws. More importantly, the means to generate these equations using quaternion operators has been displayed. This approach looks independent from the usual method which relies on an antisymmetric 2-rank field tensor and a U(1) connection.

2 Electromagnetic field gauges

A gauge is a measure of distance. Gauges are often chosen to make solving a particular problem easier. A few are well known: the Coulomb gauge for classical electromagnetism, the Lorenz gauge which makes electromagnetism look like a simple harmonic oscillator, and the gauge invariant form which is used in the Maxwell equations. In all these cases, the E and B field is the same, only the way it is measured is different. In this notebook, these are all generated using a differential quaternion operator and a quaternion electromagnetic potential.

The Field Tensor F in Different Gauges

The anti-symmetric 2-rank electromagnetic field tensor F has 3 properties: its trace is zero, it is antisymmetric, and it contains all the components of the E and B fields. The field used in deriving the Maxwell equations had the same information written as a quaternion:

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\phi, -\vec{A}\right) - \left(\phi, \vec{A}\right)\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) = \left(0, -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\nabla} X \vec{A}\right)$$

What makes this form gauge-invariant, so no matter what the choice of gauge (involving dphi/dt and Del.A), the resulting equation is identical? It is the work of the zero! Whatever the scalar field is in the first term of the generator gets subtracted away in the second term.

A mathematical aside: a friend of mine calls this a ”conjugator”. The well-known commutator involves commuting two terms and then subtracting them from the starting terms. In this case, the two terms were conjugated and then subtracted from the original. Any quaternion expression that gets acted on by a conjugator results in a 0 scalar and a 3-vector. An anti-conjugator does the opposite task. By adding together something with its conjugate, only the scalar remains. The conjugator will be used often here.

Generating the field tensor F in the Lorenz gauge starting from the gauge-invariant from involves swapping the fields in the following way:

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\frac{\left(\phi, \vec{A}\right) + \left(\phi, -\vec{A}\right)}{2}\right) - \left(\frac{\left(\phi, \vec{A}\right) - \left(\phi, -\vec{A}\right)}{2}\right)\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) =$$

$$= \left(\frac{\partial \phi}{\partial t} + \vec{\nabla}.\vec{A}, -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\nabla} X \vec{A}\right)$$

This looks more complicated than it is. The first term of the generator involves the scalar field only, (phi, 0), and the second term involves the 3-vector field only, (0, A).

The field tensor F in the Coulomb gauge is generated by subtracting away the divergence of A, which explains why the second and third terms involve only A, even though Del.A is zero :-)

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\phi, -\vec{A}\right) +$$

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\frac{\left(\phi, \vec{A}\right) - \left(\phi, -\vec{A}\right)}{4}\right) + \left(\frac{\left(\phi, -\vec{A}\right) - \left(\phi, \vec{A}\right)}{4}\right)\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) =$$

$$= \left(\frac{\partial \phi}{\partial t}, -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\nabla} X \vec{A}\right)$$

The field tensor F in the temporal gauge is quite similar to the Coulomb gauge, but some of the signs have changed to target the dphi/dt term.

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\phi, -\vec{A}\right) -$$

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\frac{\left(\phi, \vec{A}\right) + \left(\phi, -\vec{A}\right)}{4}\right) - \left(\frac{\left(\phi, -\vec{A}\right) + \left(\phi, \vec{A}\right)}{4}\right)\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) =$$

$$= \left(-\vec{\nabla}.\vec{A}, -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\nabla} X \vec{A}\right)$$

What is the simplest expression that all of these generator share? I call it the field tensor F in the light gauge:

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\phi, -\vec{A}\right) = \left(\frac{\partial \phi}{\partial t} - \vec{\nabla}.\vec{A}, -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\nabla} X \vec{A}\right)$$

The light gauge is one sign different from the Lorenz gauge, but its generator is a simple as it gets.

Implications

In the quaternion representation, the gauge is a scalar generated in such a way as to not alter the 3-vector. In a lists of gauges in graduate-level quantum field theory written by Kaku, the light gauge did not make the list of the top 6 gauges. There is a reason for this. Gauges are presented as a choice for a physicist to make. The most interesting gauges have to do with a long-running popularity contest. The relationship between gauges is guessed, not written explicitly as was done here. The term that did not make the cut stands out. Perhaps some of the technical issues in quantum field theory might be tackled in this gauge using quaternions.

3 The Maxwell Equations in the Light Gauge: QED?

What makes a theory non-classical? Use an operational definition: a classical approach neatly separates the scalar and vector terms of a quaternion. Recall how the electric field was defined (where {A, B} is the even or symmetric product over 2, and [A, B] is the odd, antisymmetric product over two or cross product).

$$\mathbf{e} = \text{vector}\left(\text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right)\right) = \left(0, -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$\mathbf{B} = \text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(\phi, \vec{A}\right)\right) = \left(0, \vec{\nabla} \times \vec{A}\right)$$

The scalar information is explicitly discarded from the E field quaternion. In this notebook, the scalar field that arises will be examined and shown to be the field which gives rise to gauge symmetry. The commutators and anticommutators of this scalar and vector field do not alter the homogeneous terms of the Maxwell equations, but may explain why light is a quantized, transverse wave.

The E and B Fields, and the Gauge with No Name

In the previous notebook, the electric field was generated differently from the magnetic field, since the scalar field was discard. This time that will not be done.

$$\mathbf{e} = \text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right) = \left(\frac{\partial \phi}{\partial t} - \vec{\nabla} \cdot \vec{A}, -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right)$$

$$\mathbf{B} = \text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(\phi, \vec{A}\right)\right) = \left(0, \vec{\nabla} \times \vec{A}\right)$$

What is the name of the scalar field, d phi/dt - Del.A which looks like some sort of gauge? It is not the Lorenz or Landau gauge which has a plus sign between the two. It is none of the popular gauges: Coulomb (Del.A = 0), axial (Az = 0), temporal (phi = 0), Feynman, unitary...

[special note: I am now testing the interpretation that this gauge constitutes the gravitational field. See the section on Einstein's Vision]

The standard definition of a gauge starts with an arbitrary scalar function psi. The following substitutions do not effect the resulting equations.

$$\phi -> \phi' = \phi - \frac{\partial \psi}{\partial t}$$

$$\vec{A} -> \vec{A}' = \vec{A} + \vec{\nabla}\psi$$

This can be written as one quaternion transformation.

$$\left(\phi, \vec{A}\right) \longrightarrow \left(\phi', \vec{A}'\right) = \left(\phi, \vec{A}\right) + \left(-\frac{\partial \psi}{\partial t}, \vec{\nabla}\psi\right)$$

The goal here is to find an arbitrary scalar and a 3-vector that does the same work as the scalar function psi. Let

$$\mathbf{p} = -\frac{\partial \psi}{\partial t} \quad \text{and} \quad \vec{\alpha} = \vec{\nabla}\psi$$

Look at how the gauge symmetry changes by taking its derivative.

$$\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(-\frac{\partial \psi}{\partial t}, \vec{\nabla}\psi\right) =$$

$$\left(-\vec{\nabla} \cdot \vec{\nabla}\psi - \frac{\partial^2 \psi}{\partial t^2}, \vec{\nabla} \times \vec{\nabla}\psi - \vec{\nabla}\frac{\partial \psi}{\partial t} + \vec{\nabla}\frac{\partial \psi}{\partial t}\right) = \left(\frac{\partial \mathbf{p}}{\partial t} - \vec{\nabla} \cdot \vec{\alpha}, 0\right)$$

This is the gauge with no name! Call it the "light gauge". That name was chosen because if the rate of change in the scalar potential phi is equal to the spatial change of the 3-vector potential A as should be the case for a photon, the distance is zero.

The Maxwell Equations in the Light Gauge

The homogeneous terms of the Maxwell equations are formed from the sum of both orders of the commutator and anticommutator.

$$\text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(\phi, \vec{A}\right)\right)\right) +$$
$$\text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{even}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right)\right) =$$
$$= \left(-\vec{\nabla}\cdot\vec{\nabla}\, X\, \vec{A}, -\vec{\nabla}\, X\, \vec{\nabla}\phi\right) = \left(0, \vec{0}\right)$$

The source terms arise from of two commutators and two anticommutators. In the classical case discussed in the previous notebook, this involved a difference. Here a sum will be used because it generates a simpler differential equation.

$$\text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{odd}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \left(\phi, \vec{A}\right)\right)\right) -$$
$$\text{even}\left(\left(\frac{\partial}{\partial t}, \vec{\nabla}\right), \text{even}\left(\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right), \left(\phi, -\vec{A}\right)\right)\right) =$$
$$= \left(\frac{\partial^2\phi}{\partial t^2} + \vec{\nabla}\cdot\vec{\nabla}\phi, -\frac{\partial^2\vec{A}}{\partial t^2} + \vec{\nabla}\, X\, \left(\vec{\nabla}\, X\, \vec{A}\right) - \vec{\nabla}\vec{\nabla}\cdot\vec{A}\right)$$
$$= \left(\frac{\partial^2\phi}{\partial t^2} + \left(\vec{\nabla}\right)^2\phi, -\frac{\partial^2\vec{A}}{\partial t^2} - \left(\vec{\nabla}\right)^2\vec{A}\right) = 4\pi\left(\rho, \vec{J}\right)$$

Notice how the scalar and vector parts have neatly partitioned themselves. This is a wave equation, except that a sign is flipped. Here is the equation for a longitudinal wave like sound.

$$\frac{\partial^2\vec{w}}{\partial t^2} - \left(\vec{\nabla}\right)^2\vec{w} = 0$$

The second time derivative of w must be the same as Del^2 w. This has a solution which depends on sines and cosines (for simplicity, the details of initial and boundary conditions are skipped, and the infinite sum has been made finite).

$$\vec{w} = \sum_{n=0}^{\infty} \text{Cos}[n\pi t]\,\text{Sin}[n\pi R]$$

$$\partial_t\partial_t\vec{w} - \partial_R\partial_R\vec{w} = 0$$

Hit w with two time derivatives, and out comes -n^2 pi^2 w. Take Del^2, and that creates the same results. Thus every value of n will satisfy the longitudinal wave equation.

Now to find the solution for the sum of the second time derivative and Del^2. One of the signs must be switched by doing some operation twice. Sounds like a job for i! With quaternions, the square of a normalized 3-vector equals (-1, 0), and it is i if y = z = 0 . The solution to Maxwell's equations in the light gauge is

$$\vec{w} = \sum_{n=0}^{\infty} \text{Cos}[n\pi t]\,\text{Sin}\left[n\pi R\vec{V}\right]$$

$$\text{if } \left(\vec{V}\right)^2 = -1, \text{ then } \partial_t\partial_t\vec{w} + \partial_R\partial_R\vec{w} = 0$$

Hit this two time derivatives yields -n^2 pi^2 w. Del^2 w has all of this and the normalized phase factor V^2 = (-1, 0). V acts like an imaginary phase factor that rotates the spatial component. The sum for any n is zero (the details of the solution depend on the initial and boundary conditions).

Implications

The solution to the Maxwell equations in the light gauge is a superposition of waves–each with a separate value of n–where the spatial part gets rotated by the 3D analogue of i. That is a quantized, transverse wave. That's fortunate, because light is a quantized transverse wave. The equations were generated by taking the classical Maxwell equations, and making them simpler.

4 The Lorentz Force

The Lorentz force acts on a moving charge. The covariant form of this law is, where W is work and P is momentum:

$$\left(\frac{dW}{d\tau}, \frac{d\vec{P}}{d\tau}\right) = \gamma e(\vec{\beta}.\vec{E}, \vec{E} + \vec{\beta} \times \vec{B})$$

In the classical case for a point charge, beta is zero and the E = k e/r^2, so the Lorentz force simplifies to Coulomb's law. Rewrite this in terms of the potentials phi and A.

$$\left(\frac{dW}{d\tau}, \frac{d\vec{P}}{d\tau}\right) = \gamma e\left(\beta.\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi\right), -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\beta}\, X\left(\vec{\nabla}X\vec{A}\right)\right)$$

In this notebook, I will look for a quaternion equation that can generate this covariant form of the Lorentz force in the Lorenz gauge. By using potentials and operators, it may be possible to create other laws like the Lorentz force, in particular, one for gravity.

A Quaternion Equation for the Lorentz Force

The Lorentz force is composed of two parts. First, there is the E and B fields. Generate those just as was done for the Maxwell equations

$$\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(\phi, \vec{A}\right) = \left(\frac{\partial\phi}{\partial t} - \vec{\nabla}\cdot\vec{A}, \frac{\partial\vec{A}}{\partial t} + \vec{\nabla}\phi + \vec{\nabla}X\vec{A}\right)$$

Another component is the 4-velocity

$$V = \left(\gamma, \gamma\vec{\beta}\right)$$

Multiplying these two terms together creates thirteen terms, only 5 of whom belong to the Lorentz force. That should not be surprising since a bit of algebra was needed to select only the covariant terms that appear in the Maxwell equations. After some searching, I found the combination of terms required to generate the Lorentz force.

$$\left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\left(\phi, -\vec{A}\right)\left(\gamma, -\gamma\vec{\beta}\right) - \left(\gamma, -\gamma\vec{\beta}\right)\left(\frac{\partial}{\partial t}, \vec{\nabla}\right)\left(\phi, \vec{A}\right) =$$

$$= \gamma\left(\vec{\beta}\cdot\left(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi\right), -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}\phi + \vec{\beta}\, X\left(\vec{\nabla}X\vec{A}\right)\right) = \gamma e\left(\vec{\beta}.\vec{E}, \vec{E} + \vec{\beta} \times \vec{B}\right)$$

This combination of differential quaternion operator, quaternion potential and quaternion 4-velocity generates the covariant form of the Lorentz operator in the Lorenz gauge, minus a factor of the charge e which operates as a scalar multiplier.

Implications

By writing the covariant form of the Lorentz force as an operator acting on a potential, it may be possible to create other laws like the Lorentz force. For point sources in the classical limit, these new laws must have the form of Coulomb's law, F = k e e'/r^2. An obvious candidate is Newton's law of gravity, F = - G m m'/r^2. This would require a different type of scalar potential, one that always had the same sign.

5 The Stress Tensor of the Electromagnetic Field

I will outline a way to generate the terms of the symmetric 2-rank stress-momentum tensor of an electromagnetic field using quaternions. This method may provide some insight into what information the stress tensor contains.

Any equation written with 4-vectors can be rewritten with quaternions. A straight translation of terms could probably be automated with a computer program. What is more interesting is when an equation is generated by the product of operators acting on quaternion fields. I have found that generator equations often yield useful insights.

A tensor is a bookkeeping device designed to keep together elements that transform in a similar way. People can choose alternative bookkeeping systems, so long as the tensor behaves the same way under transformations. Using the terms as defined in ”The classical theory of fields” by Landau and Lifshitz, the antisymmetric 2-rank field tensor F is used to generate the stress tensor T

$$T^{ik} = \frac{1}{4\pi}\left(-F^{iL}F^{k}{}_{L} + \frac{1}{4}\delta^{ik}F_{LM}F^{LM}\right)$$

I have a practical sense of an E field (the stuff that makes my hair stand on end) and a B field (the invisible hand directing a compass), but have little sense of the field tensor F, a particular combination of the other two. Therefore, express the stress tensor T in terms of the E and B fields only:

$$T^{ik} = \begin{pmatrix} W & Sx & Sy & Sz \\ Sx & mxx & mxy & myz \\ Sy & myx & myy & myz \\ Sz & mzx & mzy & mzz \end{pmatrix}$$

$$W = \frac{1}{8\pi}\left(\left(\vec{E}\right)^2 + \left(\vec{B}\right)^2\right)$$

$$Sa = \frac{1}{4\pi}\left(\vec{E}\,x\vec{B}\right)$$

$$mab = \frac{1}{4\pi}\left(-Ea\,Eb - Ba\,Bb + 0.5\,\delta_{ab}\left(\left(\vec{E}\right)^2 + \left(\vec{B}\right)^2\right)\right)$$

Together, the energy density(W), Poynting's vector (Sa) and the Maxwell stress tensor (m_ab) are all the components of the stress tensor of the electromagnetic field.

Generating a Symmetric 2-Tensor Using Quaternions

How should one rationally go about to find a generator equation that creates these terms instead of using the month-long hunt-and-peck technique actually used? Everything is symmetric, so use the symmetric product:

$$even(q, q') = \frac{q\,q' + q'\,q}{2} = \left(t\,t' - \vec{X}.\vec{X}',\ t\vec{X}' + \vec{X}\,t'\right)$$

The fields E and B are kept separate except for the cross product in the Poynting vector. Individual directions of a field can be selected by using a unit vector Ua:

$$even\left(\vec{E},\ Ux\right) = (-Ex,\ 0)\ \text{ where } Ux = (0, 1, 0, 0)$$

The following double sum generates all the terms of the stress tensor:

$$T^{ik} = \Sigma_{a=x}^{y,z}\ \Sigma_{b=x}^{y,z}\ \frac{1}{4\pi}\left(\left(\frac{even(Ua,\ Ub)}{3} - 1\right)\frac{((0,\ e)^2 + (0,\ B)^2)}{2} -\right.$$

$$- even(e,\ Ua)\,even(e,\ Ub) - even(B,\ Ua)\,even(B,\ Ub) -$$

$$- even(odd(e,\ B),\ Ua) - even(odd(e,\ B),\ Ub) =$$

$$= (-Ex\,Ey - Ex\,Ez - Ey\,Ez - Bx\,By - Bx\,Bz - By\,Bz$$
$$+ Ey\,Bz - Ez\,By + Ez\,Bx - Ex\,Bz + Ex\,By - Ey\,Bx,\ 0)/2\pi$$

The first line generates the energy density W, and part of the +0.5 delta(a, b)(Eˆ2 + Bˆ2) term of the Maxwell stress tensor. The rest of that tensor is generated by the second line. The third line creates the Poynting vector. Using quaternions, the net sum of these terms ends up in the scalar.

Does the generator equation have the correct properties? Switching the order of Ua and Ub leaves T unchanged, so it is symmetric. Check the trace, when Ua = Ub

$$\mathbf{trace}(T^{ik}) =$$

$$= \Sigma_{a=x}^{y,z} \frac{1}{4\pi}\left(\left(\frac{\mathbf{even}(Ua, Ua)}{3} - 1\right)\frac{((0, e)^2 + (0, B)^2)}{2} - \mathbf{even}(e, Ua)^2 - \mathbf{even}(B, Ua)^2\right) = 0$$

The trace equals zero, as it should.

The generator is composed of three parts that have different dependencies on the unit vectors: those terms that involve Ua and Ub, those that involve Ua or Ub, and those that involve neither. These are the Maxwell stress tensor, the Poynting vector and the energy density respectively. Changing the basis vectors Ua and Ub will effect these three components differently.

Implications

So what does the stress tensor represent? It looks like every combination of the 3-vectors E and B that avoids quadratics (like Exˆ2) and over-counting cross terms. I like what I will call the ”net” stress quaternion:

$$\mathbf{net}(T^{ik}) =$$

$$= (-Ex\,Ey - Ex\,Ez - Ey\,Ez - Bx\,By - Bx\,Bz - By\,Bz + Ey\,Bz - Ez\,By + Ez\,Bx - Ex\,Bz + Ex\,By - Ey\,Bx, 0)/2\pi$$

This has the same properties as an stress tensor. Since the vector is zero, it commutes with any other quaternion (this may be a reason it is so useful). Switching x terms for y terms would flip the signs of the terms produced by the Poynting vector as required, but not the others. There are no terms of the form Exˆ2, which is equivalent to the statement that the trace of the tensor is zero.

On a personal note, I never thought I would understand what a symmetric 2-rank tensor was, even though I listen in on a discussion of the topic. Yes, I could nod along with the algebra, but without any sense of F, it felt hollow. Now that I have a generator and a net quaternion expression, it looks quite elegant and straightforward to me.