P&S Problem 2.2 The Klein-Gordon Equation

2.2 (a) From the action, find the Klein-Gordon equation

1. Start from the action:

   $$S = \int dx^4 \, (\partial_\mu \phi^* \, \partial^\mu \phi - (\tfrac{mc}{h})^2 \phi^* \phi)\,.$$

2. Select the Lagrange density, writing out all the components:

   $$\mathfrak{L} = \partial_\mu \phi^* \, \partial^\mu \phi - (\tfrac{mc}{h})^2 \phi^* \phi = \frac{\partial \phi^*}{c\partial t}\,\frac{\partial \phi}{c\partial t} - \frac{\partial \phi^*}{\partial x}\,\frac{\partial \phi}{\partial x} - \frac{\partial \phi^*}{\partial y}\,\frac{\partial \phi}{\partial y} - \frac{\partial \phi^*}{\partial z}\,\frac{\partial \phi}{\partial z} - (\tfrac{mc}{h})^2 \, \phi^* \, \phi\,.$$

3. Calculate the connonical momentum densities π and π^* conjugate to ϕ and ϕ^*:

   $$\pi = \frac{\partial \mathfrak{L}}{\partial(\frac{\partial \phi}{c\partial t})} = \frac{\partial \phi^*}{c\partial t} \qquad \pi^* = \frac{\partial \mathfrak{L}}{\partial(\frac{\partial \phi^*}{c\partial t})} = \frac{\partial \phi}{c\partial t}\,.$$

4. Write out the Hamiltonian:

   $$H = \int dx^3 \, (\pi \, \frac{\partial \phi}{\partial t} + \pi^* \, \frac{\partial \phi^*}{\partial t} - \frac{\partial \phi^*}{c\partial t}\,\frac{\partial \phi}{c\partial t} + \frac{\partial \phi^*}{\partial x}\,\frac{\partial \phi}{\partial x} + \frac{\partial \phi^*}{\partial y}\,\frac{\partial \phi}{\partial y} + \frac{\partial \phi^*}{\partial z}\,\frac{\partial \phi}{\partial z} + (\tfrac{mc}{h})^2 \, \phi^* \, \phi$$

   $$= \int dx^3 \, (\pi \, \pi^* + \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi + (\tfrac{\mathrm{mc}}{h})^2 \, \phi^* \, \phi)\,.$$

5. Calculate the commutators:

   $$[\phi, \pi] = [\phi, \tfrac{\partial \phi^*}{c\partial t}] = i \, \tfrac{h}{c} \, \delta^3 \qquad [\phi^*, \pi^*] = [\phi^*, \tfrac{\partial \phi}{c\partial t}] = i \, \tfrac{h}{c} \, \delta^3$$

   $$[\phi, \phi] = [\pi, \pi] = [\phi, \phi^*] = [\pi, \pi^*] = [\phi^*, \phi^*] = [\pi^*, \pi^*] = 0\,.$$

6. The Heisenberg equations of motion:

   $$i \, \frac{\partial \phi}{c\partial t} = [\phi, \tfrac{c}{h} H] \qquad i \, \frac{\partial \pi}{c\partial t} = [\tfrac{c}{h} H, \pi] \qquad i \, \frac{\partial \phi^*}{c\partial t} = [\phi^*, \tfrac{c}{h} H] \qquad i \, \frac{\partial \pi^*}{c\partial t} = [\tfrac{c}{h} H, \pi^*]$$

7. Plug in non-zero part of the Hamiltonian into two equations of motion:

   a) $i \, \frac{\partial \phi}{c\partial t} = \frac{c}{h} \int dx^3 \, [\phi, \pi] \, \pi^* = \frac{c}{h} \int dx^3 \, i \, \frac{h}{c} \, \delta^3 \, \pi^* = i \, \pi^*$

   b) By the chain rule, $\vec{\nabla}(\phi^* \, \vec{\nabla} \phi) = \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi + \phi^* \, \nabla^2 \, \phi$.

   $$i \, \frac{\partial \pi^*}{c\partial t} = \frac{c}{h} \int dx^3 \, [\pi^*, \phi^*] (- \vec{\nabla}^2 + (\tfrac{\mathrm{mc}}{h})^2) \, \phi = \frac{c}{h} \int dx^3 \, (- i) \frac{h}{c} \, \delta^3 (- \nabla^2 + (\tfrac{mc}{h})^2) \, \phi$$

   $$= - \, i \, (- \nabla^2 + (\tfrac{mc}{h})^2) \, \phi\,.$$

8. Take the derivative of 7a), notice the connection to 7b):

   $$\frac{\partial}{c\partial t} \, i \, \frac{\partial \phi}{c\partial t} = i \, \frac{\partial^2 \, \phi}{c^2 \, \partial t^2} = i \, \frac{\partial \pi^*}{c\partial t} = - \, i \, (- \nabla^2 + (\tfrac{mc}{h})^2) \, \phi$$

   $$(\frac{\partial^2}{c^2 \, \partial t^2} - \nabla^2 + (\tfrac{mc}{h})^2) \, \phi = 0\,.$$

   This is the Klein-Gordon equation.

2.2 (b) Diagonalize with creation/annihilation operators

1. Start with the Hamiltonian:

   $H = \int dx^3 \, (\pi \, \pi^* + \vec{\nabla} \phi^* \cdot \vec{\nabla} \phi + (\frac{\mathrm{mc}}{h})^2 \, \phi^* \, \phi)\,.$

2. Transform the Hamiltonian to the Fourier space:

   $H = \int \, \frac{dp^3}{(2\pi h)^3} \, ((\frac{h}{c})^2 \, \frac{\partial \varphi^*}{c \partial t} \, \frac{\partial \varphi}{c \partial t} + \frac{|\boldsymbol{p}|}{c} \, \varphi^* \, \frac{|\boldsymbol{p}|}{c} \, \varphi + m \varphi^* m \varphi)\,,$

   where $\varphi = \frac{h}{c} \, \frac{1}{\phi} \, e^{\frac{i}{h}(E t - \boldsymbol{p} \cdot \boldsymbol{x})}$ and $\varphi = \frac{h}{c} \, \frac{1}{\phi^*} \, e^{-\frac{i}{h}(E t - \boldsymbol{p} \cdot \boldsymbol{x})}$, units of $\sqrt{\frac{L^3}{m}}$

3. Diagonalize by expressing the integrand as a complex product:

   $H = \int \, \frac{dp^3}{(2\pi h)^3} \, ((i \, \frac{h}{c} \, \frac{\partial}{c \partial t} + \sqrt{\frac{|\boldsymbol{p}|}{c} + m^2}) \, \varphi^* \, (- i \, \frac{h}{c} \, \frac{\partial}{c \partial t} + \sqrt{\frac{|\boldsymbol{p}|}{c} + m^2}) \, \varphi)\,,$

   true if and only if $\Pi \, \varphi - \varphi^* \, \Pi^*$ is an integration constant undert the variation in the action.

4. Rewrite with the frequency $\omega = \sqrt{\frac{|\boldsymbol{p}|}{c} + m^2}$ and Π, $\sqrt{\frac{L}{m}}$.

   $H = \int \, \frac{dp^3}{(2\pi h)^3} \, ((i \, \frac{h}{c} \, \Pi + \omega \, \varphi^*) \, (- i \, \frac{h}{c} \, \Pi^* + \omega \varphi)\,.$

5. Guess at forms for the creation a and annihilation a^1 operators, $\sqrt{m L^3}$:

   $\varphi = (a + a^1)/\omega \qquad \varphi^* = (a - a^1)/\omega \qquad \Pi = \frac{c}{h} \, (a - i \, a^1) \qquad \Pi^* = \frac{c}{h} \, (a + i \, a^1)\ .$

6. Substitute back into the Hamiltonian:

   $H = \int \, \frac{dp^3}{(2\pi h)^3} \, ((i \, (a - i \, a^1) + a - a^1)(- i \, (a + i \, a^1) + a + a^1) = \int \, \frac{dp^3}{(2\pi h)^3} \, (2 a^2 + (1 + i) a a^1\ .$

7. Try different forms to eliminate the creation operator a:

   $\varphi = (a + a^1)/\omega \qquad \varphi^* = (a - a^1)/\omega \qquad \Pi = \frac{h}{c} \, (i \, a + a^1) \qquad \Pi^* = \frac{h}{c} \, (- i \, a - a^1)\ .$

8. Substitute back into the Hamiltonian:

   $H = \int \, \frac{dp^3}{(2\pi h)^3} \, ((i \, (i a + a^1) + a - a^1)(- i \, (- i \, a - a^1) + a + a^1) = \int \, \frac{dp^3}{(2\pi h)^3} \, (- 2) \, a^{1^2}\ .$

2.2 (c) The conserved charges

1. Start from the conserved charge needed in the diagonalization step (b) 3:

   $$Q=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}\frac{\omega h}{c}(\Pi\,\varphi-\varphi^*\,\Pi^*).$$

2. Substitute in the operators from (b) 5:

   $$Q=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}((a-i\,a^\dagger)(a+a^\dagger)-(a-a^\dagger)(a+i\,a^\dagger))$$

   $$=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}(a^2-i\,a^\dagger\,a+a\,a^\dagger-i\,a^{\dagger 2}-a^2+a^\dagger\,a-i\,a\,a^\dagger+i\,a^{\dagger 2})$$

   $$=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}2(1-i)(a\,a^\dagger+a^\dagger\,a)$$

3. Substitute in the operators from (b) 7:

   $$Q=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}((i\,a+a^\dagger)(a+a^\dagger)-(a-a^\dagger)(-i\,a-a^\dagger))$$

   $$=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}(i\,a^2+a^\dagger\,a+i\,a\,a^\dagger+a^{\dagger 2}+i\,a^2-i\,a^\dagger\,a+a\,a^\dagger-a^{\dagger 2})$$

   $$=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}((1+i)\,a\,a^\dagger+(1-i)\,a^\dagger\,a)=\sqrt{G}\int\frac{dp^3}{(2\pi h)^3}(a\,a^\dagger+a^\dagger\,a+i)$$

2.2 (d) Two complex Klein-Gordon fields

The question asks to look at the conserved charges for two complex fields using Pauli sigma matrices. One needs to diagonalize the Hamiltonian as in step (b) 3, roughly involving these four products:

$$(\phi_1^*+i\,\pi_1)(\phi_1-i\,\pi_1^*),\ (\phi_1^*+\sigma^1\,\pi_2)(\phi_2-\sigma^1\,\pi_1^*),\ (\phi_2^*+\sigma^2\,\pi_1)(\phi_1-\sigma^2\,\pi_2^*),\ (\phi_2^*+\sigma^3\pi_2)(\phi_2-\sigma^3\,\pi_2^*)$$

The conserved charges arise from the cross terms:

$$i(\pi_1\phi_1-\phi_1^*\,\pi_1^*),\ \sigma^1(\pi_2\,\phi_2-\phi_1^*\,\pi_1^*),\ \sigma^2(\pi_1\,\phi_1-\phi_2^*\,\pi_2^*),\ \sigma^3(\pi_2\,\phi_2-\phi_2^*\,\pi_2^*).$$

The problem here is that i cannot be independent of the Pauli matrices. The Pauli matrices are very similar to the quaternions, the difference being that the latter is a division algebra. There are only three independent imaginary basis vectors for quaternions, not four, often represented as $i, j,$ and k. The four currents are not independent because two of these fields must lie in the same complex plane.